CERTAIN CONTINUOUS DEFORMATIONS OF SURFACES APPLICABLE TO THE QUADRICS*

BY

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Introduction.

A rectilinear congruence which possesses the property that the asymptotic lines correspond on the focal surfaces is called a W-congruence. Either focal surface of such a congruence admits an infinitesimal deformation such that the direction of deformation at a point is parallel to the normal to the other surface at the corresponding point. BIANCHI† has discovered certain W-congruences whose two focal surfaces are applicable to the same quadric Q; such a congruence may be looked upon as a transformation B_k of one focal surface into the other. We have therefrom infinitesimal deformations of surfaces applicable to quadrics. It is the purpose of this paper to show that it is possible to establish with the aid of these infinitesimal deformations the equations of a continuous deformation of such surfaces and to obtain in intrinsic form the equations of a family of surfaces each of which is a continuous deform of the others and is applicable to a quadric. A family of this sort we call a *system* (Q).‡

The paper is divided into two parts which deal respectively with the cases where Q is a paraboloid or a central quadric. In § 1 are set down certain equations and identities given by BIANCHI and others derived from them. These are applied in § 2 to the expression in analytical form of the infinitesimal deformation of a surface S, applicable to a real hyperbolic paraboloid, which is determined by a transformation B_k of S. These results enable us to find in § 3 that the intrinsic determination of a surface S reduces to the integration of two partial differential equations.

* Presented to the Society at Cleveland, December 31, 1912. This paper was also presented in part at the Fifth International Congress of Mathematicians in Cambridge, England August 22–28, 1912, and an abstract is printed in the Proceedings of the Congress.

† *Lezioni di Geometria Differenziale,* vol. III, Pisa (1909). Hereafter a reference to this volume will be of the form B, p. —.

‡ Analogous families of pseudospherical surfaces have been considered by BIANCHI, *Sopra una classe di deformazioni continue delle superficie pseudosferiche,* Annali di Matematica, ser. 3, vol. 18 (1911), pp. 1–67.

In §§ 4, 5 we derive the integrable system of differential equations which determine a system (Q) of non-ruled surfaces S, and in § 6 the similar question is handled for systems (Q) of ruled surfaces.

Since the correspondence between a surface S and a transform established by the associated W-congruence is not also the correspondence of applicability of these surfaces, it is necessary to develop in § 7 certain transformed equations referring to the latter type of correspondence. These are applied in § 8 to give an analytical proof of the reciprocal character of a transformation B_k.

In setting up a system (Q) we have associated with each surface S of the system a surface $\bar{S}$, which arises from S by a transformation B_k. We say that these surfaces form *the system conjugate* to the given one. In § 9 we find that the conjugate system is itself a system (Q) whose conjugate system is the given one. In § 10 the question of generalized transformations B_k from a system (Q) into others of the same sort is investigated.

In §§ 11, 12, 13, 14 we give in condensed form similar equations and results for surfaces S applicable to the real hyperboloid of one sheet. In order to facilitate comparison of analogous equations and identities, we have given them the same numbers in the two parts of the paper.

In § 15 we show by what change of variables and constants it is possible to transform the equations and identities so as to establish the existence of systems (Q) of pseudospherical surfaces, as found by BIANCHI.*

The closing section deals with systems (Q) of ruled surfaces applicable to the hyperboloid of revolution of one sheet and incidentally with a deformation of Bertrand curves into curves of the same kind.

PART I.

SYSTEMS (Q) OF SURFACES APPLICABLE TO A PARABOLOID.

§ 1. *Preliminary Formulas.*

If we take the equations of the paraboloid P in the form

$$x_0 = \sqrt{p}\,(u+v), \qquad y_0 = \sqrt{q}\,(u-v), \qquad z_0 = 2uv, \tag{1}$$

the generators are parametric and the first fundamental coefficients have the values

$$E = p + q + 4v^2, \qquad F = p - q + 4uv, \qquad G = p + q + 4u^2. \tag{2}$$

For the sake of brevity we define a function H by the first of the equations

$$H = \tfrac{1}{4}(EG - F^2) = p\,(u-v)^2 + q\,(u+v)^2 + pq. \tag{3}$$

* Loc. cit.

The functions E, F, G, and H satisfy the following identities which are necessary in the discussion:

$$\frac{\partial E}{\partial v}=2\frac{\partial F}{\partial u},\qquad \frac{\partial G}{\partial u}=2\frac{\partial F}{\partial v},$$

$$(4)\qquad \begin{aligned} &E\frac{\partial \log H}{\partial v}+F\frac{\partial \log H}{\partial u}-2\frac{\partial F}{\partial u}=0,\\ &F\frac{\partial \log H}{\partial v}+G\frac{\partial \log H}{\partial u}-2\frac{\partial F}{\partial v}=0,\end{aligned}$$

$$\left(\frac{\partial H}{\partial u}\right)^2+4pqE=4(p+q)H,$$

$$(5)\qquad \begin{aligned} &\frac{\partial^2 \log H}{\partial u^2}+\frac{1}{2}\left(\frac{\partial \log H}{\partial u}\right)^2=\frac{2pqE}{H^2},\\ &\frac{\partial^2 \log H}{\partial u\partial v}+\frac{1}{2}\frac{\partial \log H}{\partial u}\frac{\partial \log H}{\partial v}=-\frac{2pqF}{H^2},\\ &\frac{\partial^2 \log H}{\partial v^2}+\frac{1}{2}\left(\frac{\partial \log H}{\partial v}\right)^2=\frac{2pqG}{H^2}.\end{aligned}$$

Moreover, the Christoffel symbols have the following values in this case:

$$(6)\quad \begin{Bmatrix}11\\1\end{Bmatrix}=\begin{Bmatrix}22\\1\end{Bmatrix}=\begin{Bmatrix}11\\2\end{Bmatrix}=\begin{Bmatrix}22\\2\end{Bmatrix}=0,\quad \begin{Bmatrix}12\\1\end{Bmatrix}=\frac{1}{2}\frac{\partial \log H}{\partial v},\quad \begin{Bmatrix}12\\2\end{Bmatrix}=\frac{1}{2}\frac{\partial \log H}{\partial u};$$

and consequently the Gauss equations* assume the form

$$(7)\qquad \begin{aligned} &\frac{\partial^2 x}{\partial u^2}=DX_3,\qquad \frac{\partial^2 x}{\partial v^2}=D''X_3,\\ &\frac{\partial^2 x}{\partial u\partial v}=\frac{1}{2}\frac{\partial \log H}{\partial v}\frac{\partial x}{\partial u}+\frac{1}{2}\frac{\partial \log H}{\partial u}\frac{\partial x}{\partial v}+D'X_3,\end{aligned}$$

where D, D', D'' denote the second fundamental coefficients of a non-ruled surface S, applicable to the paraboloid, and X_3, Y_3, Z_3 are the direction-cosines of the normal to S.

The Gaussian curvature of S is found to have the value

$$(8)\qquad K=-\frac{4pq}{H^2}\equiv-\frac{1}{\rho^2},$$

ρ being thus defined. Consequently

$$(9)\qquad DD''-D'^2=-\frac{4pq}{H}.$$

* E., p. 154. A reference of this sort is to the author's *Differential Geometry*, Ginn and Co., Boston, 1909.

A transformation B_k of S is given analytically by *

$$x_1 = x + l\frac{\partial x}{\partial u} + m\frac{\partial x}{\partial v}, \tag{10}$$

where

$$l = \frac{U}{W}, \qquad m = \frac{V}{W}, \tag{11}$$

and

$$\begin{aligned} U &= 2(\sqrt{qp'} \mp \sqrt{pq'})\lambda^2 u^2 - 2(\sqrt{pq} \mp \sqrt{p'q'})\lambda u \\ &\qquad - \frac{k}{2}(\sqrt{qp'} \pm \sqrt{q'p})\lambda^2 + \tfrac{1}{2}(\sqrt{qp'} \mp \sqrt{pq'}), \\ V &= 2(\sqrt{qp'} \pm \sqrt{pq'})\lambda^2 v^2 - 2(\sqrt{qp} \pm \sqrt{p'q'})\lambda v \\ &\qquad - \frac{k}{2}(\sqrt{qp'} \mp \sqrt{q'p})\lambda^2 + \tfrac{1}{2}(\sqrt{qp'} \pm \sqrt{pq'}), \\ W &= 2\lambda[\sqrt{pq} - \sqrt{p'q}(u+v)\lambda \pm \sqrt{pq'}(u-v)\lambda], \end{aligned} \tag{12}$$

where p', q' and k are constants given by

$$p' = p - k, \qquad q' = q + k, \tag{13}$$

and λ is a function of u and v satisfying the equations

$$\begin{aligned} \frac{\partial \lambda}{\partial u} &= \frac{\sqrt{pq}}{kH}V + \frac{\epsilon}{2k\sqrt{H}}(DU + D'V), \\ \frac{\partial \lambda}{\partial v} &= \frac{\sqrt{pq}}{kH}U + \frac{\epsilon}{2k\sqrt{H}}(D'U + D''V). \end{aligned} \qquad (\epsilon = \pm 1). \tag{14}$$

When the transformation B_k is determined by the generators of the confocal quadric P_k corresponding to those of parameter u on P, the upper sign in (12) must be used and in (14) $\epsilon = +1$. When the other system on P_k is used, we take either the upper signs in (12) and $\epsilon = -1$ in (14), or the lower signs in (12) and $\epsilon = +1$ in (14).†

We define four functions as follows:

$$\begin{aligned} L_0 &= \frac{\partial l}{\partial u} + \frac{1}{2}\frac{\partial \log H}{\partial v}m + 1, & M_0 &= \frac{\partial m}{\partial u} + \frac{1}{2}\frac{\partial \log H}{\partial u}m, \\ P_0 &= \frac{\partial l}{\partial v} + \frac{1}{2}\frac{\partial \log H}{\partial v}l, & Q_0 &= \frac{\partial m}{\partial v} + \frac{1}{2}\frac{\partial \log H}{\partial u}l + 1. \end{aligned} \tag{15}$$

It is important to observe that $\partial l/\partial u, \cdots, \partial m/\partial v$, as used in (15) and

* B., p. 13.

† Cf. B., pp. 13, 88, 319.

hereafter, indicate the derivates of l and m with respect to u or v appearing explicitly in these functions and not as obtained implicitly from λ. Thus we shall always write

$$\frac{d}{du} = \frac{\partial}{\partial u} + \frac{\partial \lambda}{\partial u} \cdot \frac{\partial}{\partial \lambda}, \qquad \frac{d}{dv} = \frac{\partial}{\partial v} + \frac{\partial \lambda}{\partial v} \cdot \frac{\partial}{\partial \lambda}. \tag{16}$$

One shows that

$$L_0 l - P_0 m = 0, \qquad M_0 l - Q_0 m = 0, \tag{17}$$

and consequently (15) may be written

$$\begin{aligned} \frac{L_0}{m} = \frac{P_0}{l} = \frac{\partial \log l}{\partial v} + \frac{1}{2} \frac{\partial \log H}{\partial v}, \\ \frac{M_0}{m} = \frac{Q_0}{l} = \frac{\partial \log m}{\partial u} + \frac{1}{2} \frac{\partial \log H}{\partial u}. \end{aligned} \tag{18}$$

These results justify the following definition of a function A,

$$A = \frac{lM_0 - mL_0}{m} = \frac{lQ_0 - mP_0}{l}. \tag{19}$$

Substituting from (15) we find the following expressions for A:

$$\begin{aligned} A &= \frac{l}{m} \frac{\partial m}{\partial u} - \frac{\partial l}{\partial u} + \frac{1}{2} \frac{\partial \log H}{\partial u} l - \frac{1}{2} \frac{\partial \log H}{\partial v} m - 1, \\ &= \frac{\partial m}{\partial v} - \frac{m}{l} \frac{\partial l}{\partial v} + \frac{1}{2} \frac{\partial \log H}{\partial u} l - \frac{1}{2} \frac{\partial \log H}{\partial v} m + 1, \\ &= \frac{l}{m} \frac{\partial m}{\partial u} - \frac{m}{l} \frac{\partial l}{\partial v} + \frac{l}{2} \frac{\partial \log H}{\partial u} - \frac{m}{2} \frac{\partial \log H}{\partial v}. \end{aligned} \tag{20}$$

As an immediate result we have the identities

$$\frac{\partial l}{\partial u} - \frac{m}{l} \frac{\partial l}{\partial v} + 1 = 0, \qquad \frac{l}{m} \frac{\partial m}{\partial u} - \frac{\partial m}{\partial v} - 1 = 0. \tag{21}$$

Bianchi* has established the following important identity:

$$m \frac{\partial l}{\partial \lambda} - l \frac{\partial m}{\partial \lambda} = \frac{H}{\sqrt{pq}} \frac{k}{W} A, \tag{22}$$

or, in other form,

$$V \frac{\partial U}{\partial \lambda} - U \frac{\partial V}{\partial \lambda} = \frac{H}{\sqrt{pq}} kWA. \tag{23}$$

* B., p. 19.

In consequence of (22) we may define a function J thus

$$J = \left(\frac{kH}{\sqrt{pq}W} \frac{L_0}{m} + \frac{\partial l}{\partial \lambda} \right) m = \left(\frac{kH}{\sqrt{pq}W} \frac{M_0}{m} + \frac{\partial m}{\partial \lambda} \right) l. \tag{24}$$

If $X_1, Y_1, Z_1; X_2, Y_2, Z_2$ denote the direction-cosines of the tangents to the curves $v =$ const., $u =$ const. respectively on S, we have

$$\frac{\partial x}{\partial u} = \sqrt{E} X_1, \qquad \frac{\partial x}{\partial v} = \sqrt{G} X_2, \tag{25}$$

and similar expressions for $Y_1, \cdots Z_2$.

In consequence of (7) and (4) we have

$$\begin{aligned} \frac{\partial X_1}{\partial u} &= \frac{D}{\sqrt{E}} X_3, \quad \frac{\partial X_1}{\partial v} = \frac{1}{2E} \frac{\partial \log H}{\partial u} (-FX_1 + \sqrt{EG} X_2) + \frac{D'}{\sqrt{E}} X_3, \\ \frac{\partial X_2}{\partial u} &= \frac{1}{2G} \frac{\partial \log H}{\partial v} (\sqrt{EG} X_1 - FX_2) + \frac{D'}{\sqrt{G}} X_3, \quad \frac{\partial X_2}{\partial v} = \frac{D''}{\sqrt{G}} X_3, \end{aligned} \tag{26}$$

and the derivatives of X_3 are given by*

$$\begin{aligned} \frac{\partial X_3}{\partial u} &= \frac{FD' - GD}{4H} \sqrt{E} X_1 + \frac{FD - ED'}{4H} \sqrt{G} X_2, \\ \frac{\partial X_3}{\partial v} &= \frac{FD'' - GD'}{4H} \sqrt{E} X_1 + \frac{FD' - ED''}{4H} \sqrt{G} X_2. \end{aligned} \tag{27}$$

Moreover, the Codazzi equations for S assume the form

$$\begin{aligned} \frac{\partial D}{\partial v} - \frac{\partial D'}{\partial u} &= \frac{1}{2} \frac{\partial \log H}{\partial v} D + \frac{1}{2} \frac{\partial \log H}{\partial u} D', \\ \frac{\partial D''}{\partial u} - \frac{\partial D'}{\partial v} &= \frac{1}{2} \frac{\partial \log H}{\partial v} D' + \frac{1}{2} \frac{\partial \log H}{\partial u} D''. \end{aligned} \tag{28}$$

§ 2. *Infinitesimal Deformation of S.*

The equations of an infinitesimal deformation of S are†

$$x' = x + \epsilon \xi, \qquad y' = y + \epsilon \eta, \qquad z' = z + \epsilon \zeta, \tag{29}$$

where ϵ is an infinitesimal constant and ξ, η, ζ are functions of u and v satisfying the conditions

$$\Sigma \frac{\partial x}{\partial u} \frac{\partial \xi}{\partial u} = 0, \qquad \Sigma \frac{\partial x}{\partial v} \frac{\partial \xi}{\partial v} = 0, \qquad \Sigma \frac{\partial x}{\partial u} \frac{\partial \xi}{\partial v} + \Sigma \frac{\partial x}{\partial v} \frac{\partial \xi}{\partial u} = 0. \tag{30}$$

* E., p. 154.

† E., pp. 373, 374.

We know that if S and S_1 are the focal sheets of any W-congruence,* an infinitesimal deformation of S is given by taking ξ, η, ζ proportional to the direction-cosines of the normal to S_1 at the corresponding point, the factor of proportionality being determined by the conditions (30). A fundamental property of the transformation B_k is that S and its transform S_1, given by (10), are the focal surfaces of the W-congruence formed by the joins of corresponding points. Hence the knowledge of a transformation B_k of S leads to an infinitesimal deformation of S. We shall investigate this deformation.

The direction-cosines of the normal to S_1 are proportional to expressions of the form †

$$(31)\qquad -(Fl+Gm)\sqrt{E}X_1+(El+Fm)\sqrt{G}X_2+\frac{2\epsilon H^{\frac{3}{2}}AX_3}{\sqrt{pq}}.$$

Accordingly we put

$$(32)\qquad \xi=e^T\left[-(Fl+Gm)\sqrt{E}\,X_1+(El+Fm)\sqrt{G}X_2+\frac{2\epsilon H^{\frac{3}{2}}A}{\sqrt{pq}}X_3\right],$$

and determine T subject to the above conditions. From (32) we obtain with the aid of (26) and (27)

$$\begin{aligned}\frac{\partial\xi}{\partial u}=e^T\sqrt{E}X_1&\left[-(Fl+Gm)\left(\frac{\partial T}{\partial u}+\frac{1}{2}\frac{\partial\log H}{\partial u}\right)-\frac{\partial F}{\partial v}m-\left(F\frac{dl}{du}+G\frac{dm}{du}\right)\right.\\&\left.+\frac{\epsilon\sqrt{H}}{\sqrt{pq}}\frac{A}{2}(FD'-GD)\right]+e^T\sqrt{G}X_2\left[(El+Fm)\left(\frac{\partial T}{\partial u}+\frac{1}{2}\frac{\partial\log H}{\partial u}\right)\right.\\&\left.+\frac{\partial F}{\partial u}m+\left(E\frac{dl}{du}+F\frac{dm}{du}\right)+\epsilon\frac{\sqrt{H}}{\sqrt{pq}}\frac{A}{2}(FD-ED')\right]\\&+X_3\left[\frac{2\epsilon}{\sqrt{pq}}\frac{d}{du}(e^TH^{\frac{3}{2}}A)-(Fl+Gm)e^TD+(El+Fm)e^TD'\right],\end{aligned}$$

$$(33)\qquad\begin{aligned}\frac{\partial\xi}{\partial v}=e^T\sqrt{E}X_1&\left[-(Fl+Gm)\left(\frac{\partial T}{\partial v}+\frac{1}{2}\frac{\partial\log H}{\partial v}\right)-\frac{\partial F}{\partial v}l\right.\\&\left.-\left(F\frac{dl}{dv}+G\frac{dm}{dv}\right)+\frac{\epsilon\sqrt{H}}{\sqrt{pq}}\frac{A}{2}(FD''-GD')\right]\\&+e^T\sqrt{G}X_2\left[(El+Fm)\left(\frac{\partial T}{\partial v}+\frac{1}{2}\frac{\partial\log H}{\partial v}\right)+\frac{\partial F}{\partial u}l+\left(E\frac{dl}{dv}+F\frac{dm}{dv}\right)\right.\\&\left.+\frac{\epsilon\sqrt{H}}{\sqrt{pq}}\frac{A}{2}(FD'-ED'')\right]\\&+X_3\left[\frac{2\epsilon}{\sqrt{pq}}\frac{d}{dv}(e^TH^{\frac{3}{2}}A)-(Fl+Gm)e^TD'+(El+Fm)e^TD''\right].\end{aligned}$$

* E., p. 420.

† B., p. 44.

When these and similar values for $\partial \eta / \partial u, \cdots, \partial \zeta / \partial v$ are substituted in (30), we obtain the following equations for the determination of T:

$$(34)\qquad \begin{aligned} &\frac{\partial T}{\partial u}+\frac{\partial \log H}{\partial u}+\frac{d \log m}{du}+\frac{\epsilon\sqrt{H}}{\sqrt{pq}}\frac{A}{2}\frac{D}{m}=0, \\ &\frac{\partial T}{\partial v}+\frac{\partial \log H}{\partial v}+\frac{d \log l}{dv}-\frac{\epsilon\sqrt{H}}{\sqrt{pq}}\frac{A}{2}\frac{D''}{l}=0, \\ &l\frac{\partial T}{\partial u}-m\frac{\partial T}{\partial v}+\frac{dl}{du}-\frac{dm}{dv}-\frac{\epsilon\sqrt{H}}{\sqrt{pq}}AD'=0. \end{aligned}$$

By means of (14) the first two of equations (34) are reducible to

$$(35)\qquad \begin{aligned} &\frac{\partial T}{\partial u}+\frac{\partial \log H}{\partial u}+\frac{\partial \log m}{\partial u}+\frac{\sqrt{pq}W}{kH}\frac{\partial m}{\partial \lambda}+\frac{\epsilon W}{2k\sqrt{H}}\left(D\frac{\partial l}{\partial \lambda}+D'\frac{\partial m}{\partial \lambda}\right)=0, \\ &\frac{\partial T}{\partial v}+\frac{\partial \log H}{\partial v}+\frac{\partial \log l}{\partial v}+\frac{\sqrt{pq}W}{kH}\frac{\partial l}{\partial \lambda}+\frac{\epsilon W}{2k\sqrt{H}}\left(D'\frac{\partial l}{\partial \lambda}+D''\frac{\partial m}{\partial \lambda}\right)=0, \end{aligned}$$

and the third of (34) is found to be a consequence of the other two.

With the aid of the Codazzi equations (28) it can be shown that equations (35) satisfy the condition of integrability and consequently we have the theorem:

When a transformation B_k of a surface S applicable to the paraboloid P is known, an infinitesimal deformation of S is given by a quadrature.

§ 3. *Intrinsic Determination of Surfaces Applicable to the Paraboloid.*

With the aid of the foregoing results we are able to give an intrinsic determination of surfaces applicable to the paraboloid. To this end we observe that if the values of D, D', D'' given by (34) be substituted in equations (14) we obtain the single equation

$$(36)\qquad \begin{aligned} &l\left(\frac{\partial T}{\partial u}+\frac{\partial \log H}{\partial u}+\frac{\partial \log m}{\partial u}+\frac{\partial \log l}{\partial \lambda}\frac{\partial \lambda}{\partial u}\right) \\ &\qquad + m\left(\frac{\partial T}{\partial v}+\frac{\partial \log H}{\partial v}+\frac{\partial \log l}{\partial v}+\frac{\partial \log m}{\partial \lambda}\frac{\partial \lambda}{\partial v}\right)=0. \end{aligned}$$

In consequence of this equation the last of equations (34) may be given each of the forms

$$(37)\qquad \begin{aligned} &\frac{\partial T}{\partial v}+\frac{A}{m}+\frac{\partial \log H}{\partial v}+\frac{1}{l}\frac{\partial l}{\partial v}+\frac{1}{m}\frac{\partial m}{\partial \lambda}\frac{\partial \lambda}{\partial v}+\frac{\epsilon\sqrt{H}}{\sqrt{pq}}\frac{AD'}{2m}=0, \\ &\frac{\partial T}{\partial u}-\frac{A}{l}+\frac{\partial \log H}{\partial u}+\frac{1}{m}\frac{\partial m}{\partial u}+\frac{1}{l}\frac{\partial l}{\partial \lambda}\frac{\partial \lambda}{\partial u}-\frac{\epsilon\sqrt{H}}{\sqrt{pq}}\frac{AD'}{2l}=0. \end{aligned}$$

By means of these expressions for D' and those for D and D'' given by (34), the Gauss equation (9) may be reduced to

$$
\begin{aligned}
&\left(\frac{\partial\lambda}{\partial u}\frac{\partial T}{\partial v}-\frac{\partial\lambda}{\partial v}\frac{\partial T}{\partial u}\right)+\left(\frac{\partial\log Hl}{\partial v}+\frac{W\sqrt{pq}}{Hk}\frac{\partial l}{\partial\lambda}\right)\frac{\partial\lambda}{\partial u}\\
&\qquad-\left(\frac{\partial}{\partial u}\log Hm+\frac{W\sqrt{pq}}{Hk}\frac{\partial m}{\partial\lambda}\right)\frac{\partial\lambda}{\partial v}+\frac{\sqrt{pq}U}{Hk}\frac{\partial T}{\partial u}-\frac{\sqrt{pq}V}{Hk}\frac{\partial T}{\partial v}\\
&\qquad+\left[\frac{U\sqrt{pq}}{Hk}\left(\frac{\partial\log Hm}{\partial u}\right)-\frac{V\sqrt{pq}}{Hk}\left(\frac{\partial\log Hl}{\partial v}\right)\right]=0.
\end{aligned}\tag{38}
$$

Again, when the expression (34) for D and the first of (37) for D' are substituted in the first of the Codazzi equations (28), the resulting equation is of the same general form as (38). If the function $\frac{\partial\lambda}{\partial u}\frac{\partial T}{\partial v}-\frac{\partial\lambda}{\partial v}\frac{\partial T}{\partial u}$ be eliminated from it by means of (38), we obtain

$$
\begin{aligned}
&\left(\frac{\partial\log H}{\partial v}+\frac{\partial}{\partial v}\log\frac{A}{m}+\frac{\sqrt{pq}U}{Hk}\frac{\partial}{\partial\lambda}\log\frac{A}{m}\right)\left(\frac{\partial T}{\partial u}+\frac{\partial\log Hm}{\partial u}+\frac{\partial\log l}{\partial\lambda}\right)\\
&\qquad-\left(\frac{\partial\log H}{\partial u}+\frac{\sqrt{pq}V}{Hk}\frac{\partial}{\partial\lambda}\log\frac{A}{m}\right)\left(\frac{\partial T}{\partial v}+\frac{\partial\log Hl}{\partial v}+\frac{\partial\log m}{\partial\lambda}\right)=0.
\end{aligned}
$$

On substituting the first expression for A, as given by (20), this reduces to

$$
\begin{aligned}
&\left(\frac{\partial}{\partial v}\log\frac{l}{m}+\frac{1}{2}\frac{\partial\log H}{\partial v}+\frac{A}{m}-\frac{\sqrt{pq}W}{Hk}\frac{\partial l}{\partial\lambda}\right)\left(\frac{\partial T}{\partial u}+\frac{\partial\log Hm}{\partial u}+\frac{\partial\log l}{\partial\lambda}\right)\\
&\qquad+\left(\frac{1}{2}\frac{\partial\log H}{\partial v}-\frac{\sqrt{pq}W}{Hk}\frac{\partial m}{\partial\lambda}\right)\left(\frac{\partial T}{\partial v}+\frac{\partial}{\partial v}\log Hl+\frac{\partial\log m}{\partial\lambda}\right)=0.
\end{aligned}
$$

This in turn is reducible by means of (20) and (22) to (36).

In like manner it may be shown that the second of the Codazzi equations is satisfied when λ and T are solutions of (36) and (38). Hence we have the following theorem:

Given two equations (36), (38) *in which the functions H, l, m, U, V, W have the forms defined in* § 1; *if λ and T constitute any integral of these equations, the functions D, D', D'' given by* (34) *and* (37), *and the functions E, F, G given by* (2) *define intrinsically a surface applicable to the paraboloid. Furthermore, when a surface S has been found in this way and the coördinates x, y, z are known, a second surface of the same kind is given directly by* (10).

Later (§ 7) we shall see in what way the intrinsic equations of the second surface can be found without quadrature.

§ 4. *Continuous Deformation of S.*

We are not interested primarily in the infinitesimal deformations of S, but rather in showing that with the aid of the equations of such deformations it is possible to discover systems of surfaces arising from continuous deformations of S.

To this end we replace ξ, η, ζ in (32) by $\partial x / \partial w, \partial y / \partial w, \partial z / \partial w$, where w is a third variable, and we assume that the functions D, D', D'', λ involve this variable as well as u and v. Now we replace (32) by

$$(39) \quad \frac{\partial x}{\partial w} = e^T \left[-(Fl + Gm)\sqrt{E}X_1 + (El + Fm)\sqrt{G}X_2 + \frac{2\epsilon H^{3/2}}{\sqrt{pq}} AX_3 \right],$$

and because of (14) and (25) equations (33) may be replaced by

$$(40) \quad \begin{aligned} \frac{\partial X_1}{\partial w} &= e^T (FX_1 - \sqrt{EG}X_2) + \frac{B}{\sqrt{E}} X_3, \\ \frac{\partial X_2}{\partial w} &= e^T (\sqrt{EG}X_1 - FX_2) + \frac{C}{\sqrt{G}} X_3, \end{aligned}$$

where we have put for the sake of brevity

$$(41) \quad \begin{aligned} B &= \frac{2\epsilon}{\sqrt{pq}} \frac{d}{du} (e^T H^{3/2} A) - e^T [D(Fl + Gm) - D'(El + Fm)], \\ C &= \frac{2\epsilon}{\sqrt{pq}} \frac{d}{dv} (e^T H^{3/2} A) - e^T [D'(Fl + Gm) - D''(El + Fm)]. \end{aligned}$$

By means of the identities

$$\Sigma X_1 \frac{\partial X_3}{\partial w} + \Sigma X_3 \frac{\partial X_1}{\partial w} = 0, \qquad \Sigma X_2 \frac{\partial X_3}{\partial w} + \Sigma X_3 \frac{\partial X_2}{\partial w} = 0,$$

we find that

$$(42) \quad \frac{\partial X_3}{\partial w} = \frac{\sqrt{EG}}{4H} \left[(FX_1 - \sqrt{EG}X_2) \frac{C}{\sqrt{G}} - (\sqrt{EG}X_1 - FX_2) \frac{B}{\sqrt{E}} \right].$$

In order that these equations be consistent, it is necessary and sufficient that the following conditions be satisfied for the X's, Y's and Z's:

$$(43) \quad \frac{\partial}{\partial u}\left(\frac{\partial X_i}{\partial v}\right) = \frac{\partial}{\partial v}\left(\frac{\partial X_i}{\partial u}\right) \qquad (i = 1, 2, 3),$$

$$(44) \quad \frac{\partial}{\partial u}\left(\frac{\partial X_i}{\partial w}\right) = \frac{\partial}{\partial w}\left(\frac{\partial X_i}{\partial u}\right), \qquad \frac{\partial}{\partial v}\left(\frac{\partial X_i}{\partial w}\right) = \frac{\partial}{\partial w}\left(\frac{\partial X_i}{\partial v}\right) \qquad (i = 1, 2, 3).$$

Since equations (43) are true for each surface S, it is necessary to consider only equations (44). The first of these for $i = 1$ leads to an equation of the form

$aX_1 + bX_2 + cX_3 = 0$ and to two other equations obtained by replacing the X's by Y's and Z's respectively. These equations are equivalent to $a = b = c = 0$. By substitution these are found to reduce to the following

$$
\begin{aligned}
&\frac{D'B - DC}{4H} + e^T\left(\frac{\partial T}{\partial u} + \frac{1}{2}\frac{\partial \log H}{\partial u}\right) = 0,\\
(45)\qquad &\frac{\partial D}{\partial w} = e^T(FD - ED') + \frac{dB}{du}.
\end{aligned}
$$

In like manner the first of (44) for $i = 2$ leads to the first of (45) and to

$$
(46)\qquad \frac{\partial D'}{\partial w} = e^T(GD - FD') + \frac{dC}{du} - \frac{B}{2}\frac{\partial \log H}{\partial v} - \frac{C}{2}\frac{\partial \log H}{\partial u}.
$$

Furthermore, the second of equations (44) for $i = 1, 2$ give rise to

$$
\begin{aligned}
&\frac{BD'' - CD'}{4H} + e^T\left(\frac{\partial T}{\partial v} + \frac{1}{2}\frac{\partial \log H}{\partial v}\right) = 0,\\
(47)\qquad &\frac{\partial D'}{\partial w} = e^T(FD' - ED'') + \frac{dB}{dv} - \frac{B}{2}\frac{\partial \log H}{\partial v} - \frac{C}{2}\frac{\partial \log H}{\partial u},\\
&\frac{\partial D''}{\partial w} = e^T(GD' - FD'') + \frac{dC}{dv}.
\end{aligned}
$$

Finally, no new equations are introduced by equations (44) for $i = 3$,

In consequence of (9) the first of equations (45) and (47) are equivalent to

$$
\begin{aligned}
&B = e^T\frac{H^2}{pq}\left[D\left(\frac{\partial T}{\partial v} + \frac{1}{2}\frac{\partial \log H}{\partial v}\right) - D'\left(\frac{\partial T}{\partial u} + \frac{1}{2}\frac{\partial \log H}{\partial u}\right)\right],\\
(48)\qquad &C = e^T\frac{H^2}{pq}\left[D'\left(\frac{\partial T}{\partial v} + \frac{1}{2}\frac{\partial \log H}{\partial v}\right) - D''\left(\frac{\partial T}{\partial u} + \frac{1}{2}\frac{\partial \log H}{\partial u}\right)\right].
\end{aligned}
$$

By means of (35) these may be given the form

$$
\begin{aligned}
&B = \frac{\sqrt{H}W}{k}e^T\left[2\epsilon\frac{\partial m}{\partial \lambda} + \frac{\sqrt{H}J}{\sqrt{pq}}\left(\frac{D'}{l} - \frac{D}{m}\right)\right],\\
(49)\qquad &C = \frac{\sqrt{H}W}{k}e^T\left[-2\epsilon\frac{\partial l}{\partial \lambda} + \frac{\sqrt{H}J}{\sqrt{pq}}\left(\frac{D''}{l} - \frac{D'}{m}\right)\right].
\end{aligned}
$$

When these expressions are compared with (41), we find that in order that they be consistent it is necessary and sufficient that

$$
(50)\qquad El^2 + 2Flm + Gm^2 + \frac{H^2A^2}{pq} = \frac{2HWJ}{k\sqrt{pq}}.
$$

Later it will be shown that this is an identity.

Again, in order that the two expressions for $\partial D' / \partial w$ in (46) and (47) be equivalent, it is necessary that

$$\frac{dC}{du} - \frac{dB}{dv} + e^T (GD + ED'' - 2FD') = 0. \tag{51}$$

But when the values of B and C from (41) are substituted in this equation it is satisfied identically, by virtue of the Codazzi equations. Hence the only conditions introduced by equations (44) are

$$\begin{aligned} \frac{\partial D}{\partial w} &= e^T (FD - ED') + \frac{dB}{du}, \\ \frac{\partial D'}{\partial w} &= e^T (GD - FD') + \frac{dC}{du} - \frac{B}{2}\frac{\partial \log H}{\partial v} - \frac{C}{2}\frac{\partial \log H}{\partial u} \\ &= e^T (FD' - ED'') + \frac{dB}{dv} - \frac{B}{2}\frac{\partial \log H}{\partial v} - \frac{C}{2}\frac{\partial \log H}{\partial u}, \\ \frac{\partial D''}{\partial w} &= e^T (GD' - FD'') + \frac{dC}{dv}. \end{aligned} \tag{52}$$

Reviewing the case rapidly, we observe that in addition to (52) the equations of condition of the problem are (9), (14), (28) and (34). It must now be shown that these equations in λ, T, D, D', D'' are consistent. To this end we observe that if equations (9) and (28) be differentiated with respect to w and in the result the derivatives of D, D', D'' be replaced by the expressions (52), in which B and C are given the values (48), the resulting equations can be shown to be satisfied identically in consequence of (9) and (28) and of equations resulting from their differentiation with respect to u or v.

§ 5. *Intrinsic Determination of Systems* (Q).

Before considering the determination of functions D, D', D'', λ, T satisfying the above mentioned conditions, we call attention to the fact that these equations constitute the necessary and sufficient condition that the following system be completely integrable:

$$\frac{\partial x}{\partial u} = \sqrt{E} X_1, \qquad \frac{\partial x}{\partial v} = \sqrt{G} X_2,$$

$$\frac{\partial x}{\partial w} = e^T \left[- (Fl + Gm) \sqrt{E} X_1 + (El + Fm) \sqrt{G} X_2 + \frac{2\epsilon H^{\frac{3}{2}}}{\sqrt{pq}} A X_3 \right],$$

$$\frac{\partial X_1}{\partial u} = \frac{D}{\sqrt{E}} X_3, \qquad \frac{\partial X_1}{\partial v} = \frac{1}{2E}\frac{\partial \log H}{\partial u} (- F X_1 + \sqrt{EG} X_2) + \frac{D'}{\sqrt{E}} X_3,$$

$$\frac{\partial X_2}{\partial u} = \frac{1}{2G}\frac{\partial \log H}{\partial v}(\sqrt{EG}X_1 - FX_2) + \frac{D'}{\sqrt{G}}X_3, \qquad \frac{\partial X_2}{\partial v} = \frac{D''}{\sqrt{G}}X_3,$$

$$(53)\quad \frac{\partial X_3}{\partial u} = \frac{FD' - GD}{4H}\sqrt{E}X_1 + \frac{FD - ED'}{4H}\sqrt{G}X_2,$$

$$\frac{\partial X_3}{\partial v} = \frac{FD'' - GD'}{4H}\sqrt{E}X_1 + \frac{FD' - ED''}{4H}\sqrt{G}X_2,$$

$$\frac{\partial X_1}{\partial w} = e^T(FX_1 - \sqrt{EG}X_2) + \frac{B}{\sqrt{E}}X_3, \qquad \frac{\partial X_2}{\partial w} = e^T(\sqrt{EG}X_1 - FX_2) + \frac{C}{\sqrt{G}}X_3,$$

$$\frac{\partial X_3}{\partial w} = \frac{\sqrt{EG}}{4H}\left[(FX_1 - \sqrt{EG}X_2)\frac{C}{\sqrt{G}} - (\sqrt{EG}X_1 - FX_2)\frac{B}{\sqrt{E}}\right].$$

Hence from the general theory of triple systems of surfaces in space we know that, when these conditions are satisfied, the quadratic form

$$(54)\quad \begin{aligned} ds^2 = Edu^2 + 2Fdudv + Gdv^2 - 8Hme^T dudw + 8Hle^T dudw \\ + 4e^{2T}H\left(El^2 + 2Flm + Gm^2 + \frac{H^2A^2}{pq}\right)dw^2, \end{aligned}$$

(which arises from the first three of (53) and similar expressions in y and z) defines space referred to a triple system of surfaces, such that the surfaces $w =$ const. are applicable to one another and to the paraboloid (1).

We return now to the consideration of the system of consistent equations (9), (14), (28), (34) and (49). It was shown in § 3 that by means of (35) and (37) the functions D, D', D'' may be eliminated from (9), (14) and (28) with the result that we obtain the two equations (36) and (38). When the expressions for D, D', D'' from (35) and (37) are substituted in (49), we obtain

$$(55)\quad \begin{aligned} B &= \frac{2\epsilon\sqrt{H}We^T}{k}\left[\frac{\partial m}{\partial\lambda} + \frac{2J}{A}\left(\frac{\partial T}{\partial u} + \frac{\partial \log H}{\partial u} + \frac{\partial \log m}{\partial u} - \frac{A}{2l} + \frac{1}{2}\frac{\partial \log lm}{\partial\lambda}\frac{\partial\lambda}{\partial u}\right)\right], \\ C &= -\frac{2\epsilon\sqrt{H}We^T}{k}\left[\frac{\partial l}{\partial\lambda} - \frac{2J}{A}\left(\frac{\partial T}{\partial v} + \frac{\partial \log H}{\partial v} + \frac{\partial \log l}{\partial v} + \frac{A}{2m} + \frac{1}{2}\frac{\partial \log lm}{\partial\lambda}\frac{\partial\lambda}{\partial v}\right)\right]. \end{aligned}$$

With the aid of these expressions and the former ones equations (52) are transformable into four equations of the form

$$
(56) \qquad
\begin{aligned}
\frac{\partial^2 T}{\partial u \partial w} &= A_{11} \frac{\partial^2 T}{\partial u^2} + B_{11} \frac{\partial^2 \lambda}{\partial u^2} + A_{12} \frac{\partial^2 T}{\partial u \partial v} + B_{12} \frac{\partial^2 \lambda}{\partial u \partial v} + E_1, \\
\frac{\partial^2 \lambda}{\partial u \partial w} &= C_{11} \frac{\partial^2 T}{\partial u^2} + D_{11} \frac{\partial^2 \lambda}{\partial u^2} + C_{12} \frac{\partial^2 T}{\partial u \partial v} + D_{12} \frac{\partial^2 \lambda}{\partial u \partial v} + F_1, \\
\frac{\partial^2 T}{\partial v \partial w} &= A_{21} \frac{\partial^2 T}{\partial u \partial v} + B_{21} \frac{\partial^2 \lambda}{\partial u \partial v} + A_{22} \frac{\partial^2 T}{\partial v^2} + B_{22} \frac{\partial^2 \lambda}{\partial v^2} + E_2, \\
\frac{\partial^2 \lambda}{\partial v \partial w} &= C_{21} \frac{\partial^2 T}{\partial u \partial v} + D_{21} \frac{\partial^2 \lambda}{\partial u \partial v} + C_{22} \frac{\partial^2 T}{\partial v^2} + D_{22} \frac{\partial^2 \lambda}{\partial v^2} + F_2,
\end{aligned}
$$

where $A_{11}, \cdots, F_1, F_2$ are determinate functions of u, v, λ, T and the first derivatives of λ and T with respect to u, v and w. It is unnecessary to write down the explicit expressions for these functions, but it is important to remark that since the system of equations (9), (14), (28), (34) and (49) is consistent so also is the system (36), (38) and (56). Furthermore, when a set of values of the latter system is known, one obtains the functions D, D', D'' directly from (35) and (37).

The systems of equations is such that the formal integration reduces to the determination of power series in w, thus

$$
(57) \qquad
\begin{aligned}
\lambda &= \varphi_0 + \varphi_1 w + \varphi_2 w^2 + \cdots \\
T &= \psi_0 + \psi_1 w + \psi_2 w^2 + \cdots
\end{aligned}
$$

where the φ's and ψ's are functions of u and v. Evidently for φ_0 and ψ_0 we take a set of solutions of (36) and (38). To find φ_1 and ψ_1 we substitute the expressions (57) in (56), and put $w = 0$; this gives four equations of the form

$$
(58) \qquad \frac{\partial \varphi_1}{\partial u} = A_1, \qquad \frac{\partial \varphi_1}{\partial v} = A_2, \qquad \frac{\partial \psi_1}{\partial u} = B_1, \qquad \frac{\partial \psi_1}{\partial v} = B_2,
$$

where the functions A and B involve $\varphi_1, \psi_1, \varphi_0, \psi_0$ and the derivatives of φ_0 and ψ_0. When one has a set of functions φ_1, ψ_1, satisfying these equations, the determination of φ_2 and ψ_2 is a similar problem. In this case we differentiate equations (56) with respect to w, substitute (57) and put $w = 0$. Proceeding step by step we reduce the problem of finding the coefficients in (57) to the integration of systems of equations of the type (58). It is not our purpose to go further into the consideration of the character and domain of the solutions, but we have given sufficient indication to enable us to state that equations (36), (38), (56) possess common solutions. Hence we have the theorem:

There exist triple systems of surfaces such that the surfaces in one family are continuous deforms of one another and of the hyperbolic paraboloid.

We have thus established the existence of *systems* (Q).

Thus far we have considered only surfaces applicable to the hyperbolic paraboloid. Bianchi has derived the equations for transformations of surfaces applicable to the *elliptic paraboloid** and also to the *imaginary paraboloid*†

$$\frac{x^2}{p} + \frac{y^2}{q} = 2iz. \tag{59}$$

The changes to be made in the formulas in these two cases are such that one sees readily that by a repetition of the processes of the foregoing sections one can easily establish the existence of systems (Q) of surfaces applicable to these two types of paraboloids also. As a matter of fact we have shown elsewhere‡ that there exist systems (Q) of surfaces applicable to the paraboloid (59), the process of proof being less direct than the foregoing.

§ 6. *Systems* (Q) *of Ruled Surfaces.*

Thus far we have excluded the case where S is ruled. We consider it now, and observe that (9) may be replaced by

$$D = 0, \qquad D' = -2\frac{\sqrt{pq}}{\sqrt{H}}, \qquad D'' = 2\sqrt{H}\varphi, \tag{60}$$

where φ is independent of u, the assumption being that the lines $v = \text{const.}$ are straight. From geometrical considerations it is evident that if the transform S_1 given by (10) is to be ruled also, the upper signs in (12) must be used and $\epsilon = +1$ in (14). The latter equations reduce in this case to

$$\frac{\partial \lambda}{\partial u} = 0, \qquad \frac{\partial \lambda}{\partial v} = \frac{V\varphi(v)}{k}, \tag{61}$$

consequently λ is a function of v alone. Instead of (35) we have the consistent set

$$\begin{aligned} &\frac{\partial T}{\partial u} + \frac{\partial \log H}{\partial u} + \frac{\partial \log m}{\partial u} = 0, \\ &\frac{\partial T}{\partial v} + \frac{\partial \log H}{\partial v} + \frac{\partial \log l}{\partial v} + \frac{\varphi(v)W}{k}\frac{\partial m}{\partial \lambda} = 0, \end{aligned} \tag{62}$$

and the functions B and C which appear in (40) have the form

* B., p. 113.

† B., p. 150.

‡ *Sopra le deformazioni continue delle superficie reali applicabili sul paraboloide a parametro puramente immaginario*, Rendiconti della R. Accademia dei Lincei, vol. XXI (1912), pp. 458–462.

$$
(63)\qquad
\begin{aligned}
B &= -\frac{2H^{3/2}e^T}{\sqrt{pq}}\left(\frac{\partial \log m}{\partial u}+\frac{1}{2}\frac{\partial \log H}{\partial u}\right),\\
C &= \frac{2H^{3/2}e^T}{\sqrt{pq}}\left(\frac{\partial \log l}{\partial v}+\frac{1}{2}\frac{\partial \log H}{\partial v}-\frac{2\sqrt{pq}m\varphi}{k}\right).
\end{aligned}
$$

When these expressions and the values from (60) are substituted in (52), the first two of the latter become

$$
(64)\qquad \frac{\partial D}{\partial w}=0, \qquad \frac{\partial D'}{\partial w}=0,
$$

and the third one is reducible to

$$
(65)\qquad
\begin{aligned}
\sqrt{pq}\frac{\partial \varphi}{\partial w}+\frac{2\sqrt{pq}mHe^T}{k}\frac{\partial \varphi}{\partial v}+\varphi He^T\Bigg[\frac{\sqrt{pq}}{H}\left(F+\frac{mG}{l}\right)-\frac{H}{\sqrt{pq}}\frac{L_0}{m}\frac{A}{l}&\\
+\frac{2\sqrt{pq}}{kW}\frac{\partial V}{\partial v}\Bigg]&=0,
\end{aligned}
$$

in consequence of (5) and of the following identities which are readily established:

$$
(66)\qquad
\begin{aligned}
&\frac{\partial^2 \log m}{\partial u^2}=\left(\frac{\partial \log m}{\partial u}\right)^2, \qquad \frac{\partial^2 \log l}{\partial v^2}=\left(\frac{\partial \log l}{\partial v}\right)^2,\\
&\frac{\partial^2 \log m}{\partial u\,\partial v}=\frac{\partial^2 \log l}{\partial u\,\partial v}=\frac{\partial \log m}{\partial u}\cdot\frac{\partial \log l}{\partial v}.
\end{aligned}
$$

Since φ is at most a function of v and w, the coefficients in equation (65) must be independent of u. The first is constant and the second is evidently independent of u, in consequence of (61) and (62). As regards the coefficient of φ we observe that it may be written

$$
e^T Hm\left[\frac{\sqrt{pq}}{H}\left(\frac{F}{m}+\frac{G}{l}\right)-\frac{H}{\sqrt{pq}}\frac{L_0}{m^2}\frac{A}{l}+\frac{2\sqrt{pq}}{k}\frac{1}{V}\frac{\partial V}{\partial v}\right],
$$

and consequently we must show that

$$
\frac{\sqrt{pq}}{H}\left(\frac{F}{m}+\frac{G}{l}\right)-\frac{H}{\sqrt{pq}}\frac{L_0}{m^2}\frac{A}{l}
$$

is independent of u. In view of (19) and (50) this may be written

$$
-\frac{\sqrt{pq}}{Hm}\left[E\frac{l}{m}+F+\frac{4H}{k}+\frac{H^2}{pq}\cdot\frac{A}{m}\cdot\frac{M_0}{m}\right].
$$

If we differentiate this expression with respect to u and make use of (4) and (5), the result is reducible to

$$-\frac{H}{\sqrt{pq}}\left[H\left(\frac{\partial \log m}{\partial u}\right)^2+\frac{\partial H}{\partial u}\frac{\partial \log m}{\partial u}+p+q+\frac{\partial m}{\partial u}\cdot\frac{4}{k}pq\right],$$

and by direct substitution it is found that this expression vanishes identically.

Consider now equations (62). The first may be replaced by

$$T=\log\frac{f}{Hm}, \tag{67}$$

where f is independent of u. Substituting this value in the second, we obtain

$$\frac{\partial f}{\partial v}=\frac{V'}{V}f, \tag{68}$$

where V' denotes the derivative of V with respect to v entering explicitly and not implicitly by means of λ. Moreover, equation (65) may be written

$$\frac{\partial \varphi}{\partial w}+\frac{2}{k}f\frac{\partial \varphi}{\partial v}+f\sigma\varphi=0, \tag{69}$$

where σ is a determinate function of u, v and λ.

Equations (61), (68), (69) form a system to which the standard existence theorems apply. Hence recalling the general discussion of § 4, we have the result:

There exist systems (Q) *for which all of the surfaces* $w=$ *const. are ruled, and the determination of the intrinsic equations of such systems requires the integration of a system of partial differential equations of the first order.*

§ 7. *Change of Parameters. Fundamental Identities.*

We will now establish the fundamental identity (50), and to this end consider the transformation B_k defined by equations of the type

$$\bar{x}=x+l\frac{\partial x}{\partial u}+m\frac{\partial x}{\partial v}. \tag{70}$$

From this we have by differentiation

$$\begin{aligned}\frac{\partial \bar{x}}{\partial u}&=L_0\frac{\partial x}{\partial u}+M_0\frac{\partial x}{\partial v}+\left(\frac{\partial l}{\partial \lambda}\frac{\partial x}{\partial u}+\frac{\partial m}{\partial \lambda}\frac{\partial x}{\partial v}\right)\frac{\partial \lambda}{\partial u}+(Dl+D'm)X_3,\\ \frac{\partial \bar{x}}{\partial v}&=P_0\frac{\partial x}{\partial u}+Q_0\frac{\partial x}{\partial v}+\left(\frac{\partial l}{\partial \lambda}\frac{\partial x}{\partial u}+\frac{\partial m}{\partial \lambda}\frac{\partial x}{\partial v}\right)\frac{\partial \lambda}{\partial v}+(D'l+D''m)X_3,\end{aligned} \tag{71}$$

where L_0, M_0, P_0, Q_0 are defined by (15).

By means of (71) we can find the first fundamental coefficients of $\bar{S}$, but

as Bianchi* has remarked, these functions have not the same form as (2). He proved,† however, that it is possible to transform the parameters on $\bar{S}$ into a new system $\bar{u}$, $\bar{v}$ so that the new coefficients $\bar{E}$, $\bar{F}$, $\bar{G}$ as functions of $\bar{u}$ and $\bar{v}$ have the form (2). In fact, this change of parameters is made in accordance with the affine transformation of Ivory, whose analytic form is

$$\bar{u} = \frac{\sqrt{qp'}\,(u+v) + \sqrt{pq'}\,(u-v) - \sqrt{pq}\,(4uv+k)\lambda}{2\left[\sqrt{pq'}\,(u-v)\,\lambda - \sqrt{qp'}\,(u+v)\,\lambda + \sqrt{pq}\right]}, \tag{72}$$
$$\bar{v} = \frac{1}{2\lambda}.$$

By differentiation we have

$$\frac{d\bar{u}}{du} = 4\sqrt{pq}\lambda^2\frac{V}{W^2} + \frac{\partial u_1}{\partial\lambda}\frac{\partial\lambda}{\partial u}, \tag{73}$$
$$\frac{d\bar{u}}{dv} = 4\sqrt{pq}\lambda^2\frac{U}{W^2} + \frac{\partial u_1}{\partial\lambda}\frac{\partial\lambda}{\partial v},$$

and

$$\frac{d\bar{v}}{du} = -\frac{1}{2\lambda^2}\frac{\partial\lambda}{\partial u}, \qquad \frac{d\bar{v}}{dv} = -\frac{1}{2\lambda^2}\frac{\partial\lambda}{\partial v}. \tag{74}$$

By means of (14) we obtain

$$\delta = \frac{d(\bar{u},\bar{v})}{d(u,v)} = \frac{\epsilon\sqrt{pq}}{Hk}\,(Dl^2 - D''m^2). \tag{75}$$

From these results and the expression‡

$$\frac{\partial\bar{u}}{\partial\lambda} = -\frac{2k\lambda^2}{W^2}H, \tag{76}$$

we get from (71)

$$\frac{\partial\bar{x}}{\partial\bar{u}} = \frac{1}{4\lambda^2}\frac{W}{\sqrt{pq}}\left[\frac{L_0}{m}\frac{\partial x}{\partial u} + \frac{M_0}{m}\frac{\partial x}{\partial v} - \frac{\epsilon\sqrt{pq}}{\sqrt{H}}2X_3\right], \tag{77}$$
$$\frac{\partial\bar{x}}{\partial\bar{v}} = -2\lambda^2\left[\left(\frac{J}{m} - \frac{1}{2}\frac{kH}{W\sqrt{pq}}\frac{L_0}{m}\right)\frac{\partial x}{\partial u} + \left(\frac{J}{l} - \frac{1}{2}\frac{kH}{W\sqrt{pq}}\frac{M_0}{m}\right)\frac{\partial x}{\partial v} + \epsilon k\frac{\sqrt{H}}{W}X_3\right].$$

If for the sake of brevity we put

$$K_1 = L_0^2E + 2L_0M_0F + M_0^2G + \frac{4pqm^2}{H}, \tag{78}$$
$$K_2 = \frac{L_0E + M_0F}{m} + \frac{FL_0 + GM_0}{l},$$

* B., pp. 25, 76–79.

† B., p. 32.

‡ B., p. 35.

we have from (77) and (24)

$$(79)\qquad \begin{aligned} \bar{E} &= \frac{1}{16\lambda^4}\frac{W^2}{pqm^2}K_1, \qquad \bar{F} = -\left(\frac{J}{m}K_2 - \frac{1}{2}\frac{kH}{W\sqrt{pq}m^2}K_1\right)\frac{W}{2\sqrt{pq}}, \\ \bar{G} &= 4\lambda^4\left[(El^2 + 2Flm + Gm^2)\frac{J^2}{l^2m^2} - K_2\frac{kH}{W\sqrt{pq}}\frac{J}{m} + \frac{1}{4}\frac{k^2H^2}{W^2pqm^2}K_1\right]. \end{aligned}$$

With the aid of these values we find

$$(80)\qquad 4\bar{H} = \bar{E}\bar{G} - \bar{F}^2 = \frac{W^2}{Hl^2m^2}\left(El^2 + 2Flm + Gm^2 + \frac{A^2H^2}{pq}\right)J^2.$$

Bianchi has established* the identity

$$(81)\qquad El^2 + 2Flm + Gm^2 + \frac{1}{k^2W^2}\left(V\frac{\partial U}{\partial\lambda} - U\frac{\partial V}{\partial\lambda}\right)^2 = \frac{H\bar{H}}{pq},$$

which in consequence of (23) may be written

$$(82)\qquad El^2 + 2Flm + Gm^2 + \frac{A^2H^2}{pq} = \frac{H\bar{H}}{pq}.$$

Comparing this with (80) we see that we must have

$$J = \epsilon'\frac{2lm\sqrt{pq}}{W},$$

where $\epsilon' = \pm 1$, to be determined. From this result and (82) it follows that, if (50) is an identity, so also is

$$(83)\qquad \bar{H} = 4\epsilon'\frac{lmpq}{k}$$

and conversely. Making use of (13) and the form of $\bar{H}$ analogous to (3), we find that the preceding equation is identically satisfied when ϵ' is -1. Hence we have

$$(84)\qquad J = -\frac{2lm\sqrt{pq}}{W},$$

and the identity (50) is established. The latter may now be written, in consequence of (84),

$$(85)\qquad El^2 + 2Flm + Gm^2 + \frac{H^2A^2}{pq} = -\frac{4Hlm}{k}.$$

To these may be added the following useful identities which arise from (24) and (84)

* B., p. 46.

$$(86)\qquad \frac{\partial l}{\partial \lambda} = -\frac{\sqrt{pq}}{W}\left(\frac{kH}{pq}\frac{L_0}{m} + 2l\right), \qquad \frac{\partial m}{\partial \lambda} = -\frac{\sqrt{pq}}{W}\left(\frac{kH}{pq}\frac{M_0}{m} + 2m\right).*$$

The foregoing results enable us to find the expressions for the second fundamental coefficients of $\bar{S}$ in simple form. From (31) and (32) it follows that the direction-cosines of the normal to $\bar{S}$ are of the form

$$(87)\qquad \bar{X}_3 = Re^{-T}\xi, \quad \text{where} \quad R^{-2} = El^2 + 2Flm + Gm^2 + \frac{H^2A^2}{pq}.$$

From these we have by means of (40)

$$(88)\qquad \begin{aligned} \frac{\partial \bar{X}_3}{\partial u} &= R[\sqrt{E}(FX_1 - \sqrt{EG}X_2) + Be^{-T}X_3] + \bar{X}_3\frac{\partial}{\partial u}\log Re^{-T}, \\ \frac{\partial \bar{X}_3}{\partial v} &= R[\sqrt{G}(\sqrt{EG}X_1 - FX_2) + Ce^{-T}X_3] + \bar{X}_3\frac{\partial}{\partial v}\log Re^{-T}. \end{aligned}$$

In consequence of (14) and (86) equations (71) may be given the form

$$(89)\qquad \begin{aligned} \frac{\partial \bar{x}}{\partial u} &= \left[-\frac{2pqlm}{kH} + \frac{\epsilon}{2k\sqrt{H}}(DU + D'V)\frac{\partial l}{\partial \lambda}\right]\sqrt{E}X_1 \\ &\quad + \left[-\frac{2pqm^2}{kH} + \frac{\epsilon}{2k\sqrt{H}}(DU + D'V)\frac{\partial m}{\partial \lambda}\right]\sqrt{G}X_2 + (Dl + D'm)X_3, \\ \frac{\partial \bar{x}}{\partial v} &= \left[-\frac{2pql^2}{kH} + \frac{\epsilon}{2kH}(D'U + D''V)\frac{\partial l}{\partial \lambda}\right]\sqrt{E}X_1 \\ &\quad + \left[-\frac{2pqlm}{kH} + \frac{\epsilon}{2k\sqrt{H}}(D'U + D''V)\frac{\partial m}{\partial \lambda}\right]\sqrt{G}X_2 + (D'l + D''m)X_3. \end{aligned}$$

From these expressions we find

$$(90)\qquad \bar{D} = -\Sigma\frac{\partial \bar{X}_3}{\partial u}\frac{\partial \bar{x}}{\partial u} = \sigma D, \qquad \bar{D}' = \sigma D', \qquad \bar{D}'' = \sigma D'',$$

where

$$(91)\qquad \sigma = -\frac{2RH}{k}(Dl^2 - D''m^2).$$

Hence the asymptotic lines on S and $\bar{S}$ correspond and we have the following result, established by Bianchi in another manner:†

The surfaces S and $\bar{S}$, the latter resulting from a transformation B_k of the former, are the focal surfaces of a W-congruence formed by the lines joining corresponding points.

---

* If the other system of generators of the quadric Q_k are used, so that we take the lower signs in the expressions for U and V, it merely comes to changing the sign of every radical under which q' appears. So far as the present result goes, it is the same in both cases, as one sees readily by reviewing the above work.

† B., p. 89.

We have not written down the first fundamental coefficients E, F, G referring to the parameters u and v on $\bar{S}$, but one gets them at once from (71). Hence, when one has a surface S defined intrinsically as explained in § 3, a transform $\bar{S}$ is given intrinsically at once.

§ 8. *The Inverse of a Transformation B_k.*

Since S and $\bar{S}$ are the focal sheets of a congruence, we have

$$x = \bar{x} + \bar{l}\frac{\partial \bar{x}}{\partial \bar{u}} + \bar{m}\frac{\partial \bar{x}}{\partial \bar{v}}. \tag{92}$$

Substituting this expression in (70) and making use of (77), we obtain an equation of the form

$$a\frac{\partial x}{\partial u} + b\frac{\partial x}{\partial v} + cX_3 = 0, \tag{93}$$

where a, b, c are determinate expressions which must vanish identically because y and z also satisfy (93). These identities are

$$l + \frac{\bar{l}}{4\lambda^2}\frac{W}{\sqrt{pq}}\frac{L_0}{m} + \bar{m}\lambda^2\left(\frac{kH}{\sqrt{pq}W}\frac{L_0}{m} + \frac{4\sqrt{pq}l}{W}\right) = 0,$$

$$m + \frac{\bar{l}}{4\lambda^2}\frac{W}{\sqrt{pq}}\frac{M_0}{m} + \bar{m}\lambda^2\left(\frac{kH}{\sqrt{pq}W}\frac{M_0}{m} + \frac{4\sqrt{pq}m}{W}\right) = 0,$$

$$\frac{\bar{l}W}{4\lambda^2} + \frac{\bar{m}kH\lambda^2}{W} = 0,$$

which are equivalent to

$$\bar{l} = \frac{kH\lambda^2}{\sqrt{pq}W}, \qquad \bar{m} = -\frac{W}{4\sqrt{pq}\lambda^2}. \tag{94}$$

From these follows

$$H = -\frac{4pq\bar{l}\bar{m}}{k}, \tag{95}$$

which is analogous to (83) where $\epsilon' = -1$.

We wish to show that in fact S is a transform B_k of $\bar{S}$. To this end we put for brevity

$$A' = \sqrt{qp'} + \sqrt{pq'}, \qquad B' = \sqrt{qp'} - \sqrt{pq'},$$

with the result that (72) becomes

$$\bar{u} = \frac{A'u + B'v - \sqrt{pq}\lambda(4uv + k)}{(\sqrt{pq} - B'u\lambda - A'v\lambda)^2}.$$

If we put $v = 1/2\bar{\lambda}$ and solve the foregoing equation with respect to u, we obtain

$$u = \frac{A'\bar{u} + B'\bar{v} - \sqrt{pq}\bar{\lambda}\,(4\bar{u}\bar{v} + k)}{(\sqrt{pq} - B'\bar{u}\bar{\lambda} - A\bar{v}\bar{\lambda})^2},$$

which shows the reciprocal character of the transformation (72).

If we take

$$\bar{l} = \frac{\bar{U}}{\bar{\bar{W}}}, \qquad \bar{m} = \frac{\bar{V}}{\bar{\bar{W}}},$$

where $\bar{U}$, $\bar{V}$, $\bar{W}$ are the functions obtained on replacing u, v, λ in (12) by $\bar{u}$, $\bar{v}$, $\bar{\lambda}$, by means of the foregoing results these expressions are reducible to the form (94). Hence we have established in an analytical manner the following theorem which Bianchi* discovered by geometrical considerations:

If $\bar{S}$ is obtained from S by a transformation B_k, S may be obtained from $\bar{S}$ by another transformation $\bar{B}_k$, the inverse of the former.

§ 9. *The Conjugate System* (Q).

Suppose that we have a system (Q) and that upon each surface S of the system we effect a transformation B_k, given by

$$(96) \qquad x_1 = x + l_1 \frac{\partial x}{\partial u} + m_1 \frac{\partial x}{\partial v},$$

where l_1 and m_1 are given by replacing λ and k in (11) and (12) by λ_1 and k_1.

If (96) be differentiated with respect to w, the result is reducible by (39) and (40) to

$$(97) \qquad \begin{aligned} \frac{dx_1}{dw} = X_1 \sqrt{E} &\left\{ e^T[F(l_1 - l) + G(m_1 - m)] + \frac{\partial l_1}{\partial \lambda_1}\frac{\partial \lambda_1}{\partial w} \right\} \\ + X_2 \sqrt{G} &\left\{ -e^T[E(l_1 - l) + F(m_1 - m)] + \frac{\partial m_1}{\partial \lambda_1}\frac{\partial \lambda_1}{\partial w} \right\} \\ + X_3 &\left\{ e^T \frac{2\epsilon H^{\frac{3}{2}} A}{\sqrt{pq}} + l_1 B + m_1 C \right\}. \end{aligned}$$

It should be observed that in the functions B and C there appear l and m, but not l_1 and m_1.

From the given system (Q) we obtain by means of (96) a triple system of surfaces expressed in terms of parameters u, v, w, such that the surfaces $w =$ const. are applicable to the paraboloid. If we wish this system to be given in terms of the parameters u_1, v_1, w, where u_1 and v_1 are obtained from

* B., p. 36.

(72) by replacing λ and k by λ_1 and k_1, the direction of the curves for which both u_1 and v_1 are constant is given by

$$\frac{dx_1}{dw} - \frac{\partial x_1}{\partial u_1}\frac{\partial u_1}{\partial \lambda_1}\frac{\partial \lambda_1}{\partial w} - \frac{\partial x_1}{\partial v_1}\frac{\partial v_1}{\partial \lambda_1}\frac{\partial \lambda_1}{\partial w} \equiv \frac{\partial x_1}{\partial w}.$$

From (97), (77) and (76) we find

$$\begin{aligned}\frac{\partial x_1}{\partial w} = X_1 \sqrt{E}e^T[\,F\,(l_1 - l) + G(m_1 - m)\,] \\ \text{(98)} \qquad - X_2 \sqrt{G}e^T[\,E\,(l_1 - l) + F\,(m_1 - m)\,] \\ + X_3\left[e^T\frac{2\epsilon H^{\frac{3}{2}} A}{\sqrt{pq}} + l_1 B + m_1 C - \frac{2\epsilon_1 k_1 \sqrt{H}}{W_1}\frac{\partial \lambda_1}{\partial w}\right],\end{aligned}$$

ϵ_1 being ± 1 and not necessarily equal to ϵ (it arises from (77)).

Before considering a general system of surfaces S_1 thus defined, we will look at the surfaces $\bar{S}$ which are used in constructing the system (Q). For this particular case we have

$$l_1 = l, \quad m_1 = m, \quad \lambda_1 = \lambda, \quad k_1 = k, \quad \epsilon_1 = \epsilon, \quad u_1 = \bar{u}, \quad v_1 = \bar{v},$$

and so (98) reduces to

$$\text{(99)} \qquad \frac{\partial \bar{x}}{\partial w} = 2\left[\frac{e^T H}{k}(Dl^2 - D'' m^2) - \frac{\epsilon k \sqrt{H}}{W}\frac{\partial \lambda}{\partial w}\right]X_3.$$

Hence a tangent to a curve of parameter w is parallel to the normal to S at the corresponding point.

The parameters $\bar{u}$ and $\bar{v}$ were chosen so that the surfaces $\bar{S}$ are seen to be applicable to one another and to the paraboloid. The relation between a pair of surfaces S and $\bar{S}$ is reciprocal, as shown in § 8. The curves along which the surfaces $\bar{S}$ are deformed into one another are such that a tangent to such a curve is parallel to the corresponding normal to S. Hence if we say that the surfaces $\bar{S}$ form a system *conjugate* to the given system (Q), we have the result

The system conjugate to a given system (Q) is itself a system (Q).

In order to give further consequences of the foregoing results, we observe that in consequence of (81) we may replace (39) by

$$\text{(100)} \qquad \frac{\partial x}{\partial w} = e^T R^{-1}\bar{X}_3, \qquad R^{-2} = \frac{4H^2 \bar{H}}{pq}$$

If in like manner we put in accordance with (99)

$$\frac{\partial \bar{x}}{\partial w} = e^T \bar{R}^{-1} X_3,$$

we have from (77) and with the aid of (94)

$$\Sigma \frac{\partial \bar{x}}{\partial \bar{u}} \frac{\partial \bar{x}}{\partial w} = 4\epsilon \bar{H} \bar{m} e^{T}, \qquad \Sigma \frac{\partial \bar{x}}{\partial \bar{v}} \frac{\partial \bar{x}}{\partial w} = -4\epsilon \bar{H} \bar{l} e^{T},$$

provided that

$$\frac{1}{\bar{R}^2} = \frac{4H\bar{H}^2}{pq}, \tag{101}$$

which evidently is the analogue use of (100), and from the preceding results we know that it is true.

§ 10. *Transformations of Systems* (Q).

We return to the consideration of equations (96) and (98). The former when applied to a system (Q) leads to a triple system of surfaces such that each of the surfaces $w =$ const. is applicable to the paraboloid, by a suitable change of parameters involving λ_1. It is natural to inquire whether it is possible to determine λ_1 involving w in such a way that the transformed surfaces shall form a system (Q).

If such a transformation is possible, the function λ_1 must satisfy the equations

$$\begin{aligned} \frac{\partial \lambda_1}{\partial u} &= \frac{\sqrt{pq}}{k_1 H} V_1 + \frac{\epsilon_1}{2k_1 H}(DU_1 + D' V_1), \\ \frac{\partial \lambda_1}{\partial v} &= \frac{\sqrt{pq}}{k_1 H} U_1 + \frac{\epsilon_1}{2k_1 H}(D' U_1 + D'' V_1), \\ \frac{\partial \lambda_1}{\partial w} &= \frac{W_1 \epsilon_1}{2k_1 \sqrt{H}}(l_1 B + m_1 C) + \frac{e^T W_1}{H k_1} \pi, \end{aligned} \tag{102}$$

where π is a function to be determined.

Referring to (54) we observe that in consequence of (85) the condition

$$\Sigma \frac{\partial x}{\partial u} \frac{\partial x}{\partial w} \cdot \Sigma \frac{\partial x}{\partial v} \frac{\partial x}{\partial w} = k \Sigma \left(\frac{\partial x}{\partial w}\right)^2 \tag{103}$$

must be satisfied by a system (Q). From the results obtained in certain special cases it is probable that this condition is sufficient. Applying it to the surfaces S_1 we obtain a quadratic equation in π, namely

$$(k_1 - k)\pi^2 + P\pi + Q = 0, \tag{104}$$

where P and Q are determinate functions of $u, v, \lambda, \lambda_1, k$ and k_1.

The conditions of integrability of equations (102) lead to four equations of the form

$$
(105)\qquad
\begin{aligned}
&\frac{\partial \pi}{\partial u} + \xi_1 \pi + \eta_1 = 0, \qquad && \frac{\partial \pi}{\partial v} + \xi_2 \pi + \eta_2 = 0, \\
&\frac{\partial \pi}{\partial \lambda} + \xi_3 \pi + \eta_3 = 0, \qquad && \frac{\partial \pi}{\partial \lambda_1} + \xi_4 \pi + \eta_4 = 0,
\end{aligned}
$$

where the functions ξ_i, η_i are determinate. Hence if a function π exists which satisfies (104) and (105), it may be found by differentiation.

We have shown elsewhere* by other methods that these transformed systems (Q) exist for the case where the surfaces S are applicable to the imaginary paraboloid. Since there is no essential difference in the formulas of transformations B_k when the fundamental quadric is a hyperbolic or an imaginary paraboloid, we have good reason to believe that a function π exists satisfying (104) and (105), and that the function λ_1 given by (102) leads to a transformed system (Q). Incidentally we remark that, in the case where the surfaces S are applicable to the imaginary paraboloid, for the curves of deformation of the surfaces S_1 the tangents are parallel to the normals to surfaces $\bar{S}_1$ applicable to the same paraboloid, and each surface $\bar{S}_1$ forms with the corresponding surfaces S, $\bar{S}$, S_1 a quartern in accordance with the "theorem of permutability" which Bianchi† has established for transformations B_k.

Part II.

Systems (Q) of Surfaces Applicable to a Central Quadric.

§ 11. *Preliminary Formulas.*

In the following sections we consider systems (Q) of surfaces S applicable to the hyperboloid

$$
(1)\qquad x_0 = a\frac{1+uv}{u+v}, \qquad y_0 = b\frac{u-v}{u+v}, \qquad z_0 = c\frac{1-uv}{u+v},
$$

the parameters referring to the generators. From these we obtain

$$
(2)\qquad
\begin{aligned}
E &= [\,(a^2+c^2)\,v^4 + 2\,(c^2-a^2+2b^2)\,v^2 + a^2 + c^2\,]\,(u+v)^{-4}, \\
F &= [\,(a^2+c^2)\,u^2v^2 + (c^2-a^2)\,(u^2+v^2) - 4b^2uv + a^2 + c^2\,]\,(u+v)^{-4}, \\
G &= [\,(a^2+c^2)\,u^4 + 2\,(c^2-a^2+2b^2)\,u^2 + a^2 + c^2\,]\,(u+v)^{-4},
\end{aligned}
$$

and

$$
(3)\qquad H = \tfrac{1}{4}\,(EG - F^2) = \frac{a^2b^2(1-uv)^2 + b^2c^2(1+uv)^2 + a^2c^2(u-v)^2}{(u+v)^6}.
$$

* *Sopra le deformazioni*, etc., l. c., p. 461.

† B., p. 156–179.

It is not our purpose to repeat the details for systems (Q) now under discussion, but merely to give the fundamental equations and identities and state the results. In general there are equations and expressions analogous to all of those in Part I, and in giving certain of them we will use the same numbers as in the former part, so that the reader may compare them and fill in the gaps.

We find readily that

$$K = -\frac{a^2 b^2 c^2}{(u+v)^8 H^2} \equiv -\frac{1}{\rho^2}, \tag{8}$$

and

$$DD'' - D'^2 = -\frac{4a^2 b^2 c^2}{H\,(u+v)^8}. \tag{9}$$

A transformation B_k of S is given analytically by

$$x_1 = x + l\frac{\partial x}{\partial u} + m\frac{\partial x}{\partial v}, \tag{10}$$

with*

$$l = (u+v)\frac{U}{W}, \qquad m = (u+v)\frac{V}{W}, \tag{11}$$

the various functions being defined by

$$\begin{aligned} U &= \left(\pm\frac{ac}{a'c'} - \frac{b}{b'}\right)2u\cos\theta + \left(\pm\frac{bc}{b'c'} - \frac{a}{a'}\right)(u^2-1)\sin\theta \\ &\qquad + \left(\frac{ab}{a'b'} \mp \frac{c}{c'}\right)(u^2+1), \\ V &= \left(\mp\frac{ac}{a'c'} - \frac{b}{b'}\right)2v\cos\theta + \left(\pm\frac{bc}{b'c'} + \frac{a}{a'}\right)(v^2-1)\sin\theta \\ &\qquad + \left(\frac{ab}{a'b'} \pm \frac{c}{c'}\right)(v^2+1), \\ W &= 2\left[\pm\frac{ac}{a'c'}(u-v)\cos\theta \mp \frac{bc}{b'c'}(1+uv)\sin\theta + \frac{ab}{a'b'}(1-uv)\right], \end{aligned} \tag{12}$$

where the constants are in the relations

$$a' = \sqrt{a^2+k}, \qquad b' = \sqrt{b^2+k}, \qquad c' = \sqrt{c^2-k}, \tag{13}$$

and θ is a function of u and v such that†

$$\begin{aligned} \frac{\partial\theta}{\partial u} &= \frac{a'b'c'}{k(u+v)^2\rho}V + \epsilon\frac{a'b'c'}{2k\sqrt{abc}\sqrt{\rho}}(DU + D'V), \\ \frac{\partial\theta}{\partial v} &= \frac{a'b'c'}{k(u+v)^2\rho}U + \epsilon\frac{a'b'c'}{2k\sqrt{abc}\sqrt{\rho}}(D'U + D''V). \end{aligned} \tag{14}$$

* B., p. 49.

† B., p. 103.

Here the signs in (12) and ϵ in (14) have a significance analogous to that in Part I.

For the sake of brevity we write

$$(15)\qquad \begin{aligned} L_0 &= \frac{\partial l}{\partial u} - \frac{2}{u+v} l + \frac{1}{2}\frac{\partial \log \rho}{\partial v} m + 1, \qquad M_0 = \frac{\partial m}{\partial u} + \frac{1}{2}\frac{\partial \log \rho}{\partial u} m, \\ P_0 &= \frac{\partial l}{\partial v} + \frac{1}{2}\frac{\partial \log \rho}{\partial v} l, \qquad Q_0 = \frac{\partial m}{\partial v} - \frac{2}{u+v} m + \frac{1}{2}\frac{\partial \log \rho}{\partial u} l + 1, \end{aligned}$$

it being understood that the following notation is used:

$$(16)\qquad \frac{d}{du} = \frac{\partial}{\partial u} + \frac{\partial \theta}{\partial u} \cdot \frac{\partial}{\partial \theta}, \qquad \frac{d}{dv} = \frac{\partial}{\partial v} + \frac{\partial \theta}{\partial v} \cdot \frac{\partial}{\partial \theta}.$$

We make use also of the function A defined by

$$(19)\qquad A = \frac{lM_0 - mL_0}{m} = \frac{lQ_0 - mP_0}{l},$$

and we may show that

$$(22)\qquad A = \pm \frac{a'b'c'}{k(u+v)^2} \cdot \frac{V}{m\rho}\left(m\frac{\partial l}{\partial \theta} - l\frac{\partial m}{\partial \theta}\right).$$

We are thus enabled to define a function J by

$$(24)\qquad J = m\left(\frac{\partial l}{\partial \theta} + \frac{k\rho(u+v)^2}{a'b'c'}\frac{L_0}{V}\right) = l\left(\frac{\partial m}{\partial \theta_0} + \frac{k\rho(u+v)^2}{a'b'c'}\frac{M_0}{V}\right).$$

For the present case we have equations of the same form as (25), (27) § 1 and if in (26) § 1 we replace H by ρ we obtain equations valid for surfaces applicable to the hyperboloid.

§ 12. *Infinitesimal Deformations of S. Intrinsic Determination of Surfaces Applicable to the Hyperboloids.*

The equations

$$(29)\qquad x' = x + e\xi. \qquad y' = y + e\eta, \qquad z' = z + e\zeta$$

define an infinitesimal deformation of S, where e denotes an infinitesimal constant, and ξ, η, ζ are functions of the form*

$$(32)\qquad \xi = e^T\left[-(Fl + Gm)\sqrt{E}X_1 + (El + Fm)\sqrt{G}X_2 + \frac{2\epsilon\sqrt{abc\rho^{\frac{3}{2}}}}{(u+v)^2}AX_3\right],$$

provided that T satisfies the equations

* In this formula $\epsilon = \pm 1$.

$$
(34)\quad
\begin{aligned}
&\frac{\partial T}{\partial u}-\frac{2}{u+v}+\frac{\partial \log \rho}{\partial u}+\frac{d}{du}\log m+\frac{\epsilon\sqrt{abc}\,\rho^{\frac{3}{2}}A}{2(u+v)^2H}\frac{D}{m}=0,\\
&\frac{\partial T}{\partial v}-\frac{2}{u+v}+\frac{\partial \log \rho}{\partial v}+\frac{d}{dv}\log l-\frac{\epsilon\sqrt{abc}\,\rho^{\frac{3}{2}}A}{2(u+v)^2H}\frac{D''}{l}=0,\\
&l\left(\frac{\partial T}{\partial u}-\frac{4}{u+v}\right)-m\left(\frac{\partial T}{\partial v}-\frac{4}{u+v}\right)+\frac{dl}{du}-\frac{dm}{dv}-\frac{\epsilon\sqrt{abc}\,\rho^{\frac{3}{2}}A}{(u+v)^2H}D'=0.
\end{aligned}
$$

By means of (14) the first two of these equations are reducible to

$$
(35)\quad
\begin{aligned}
&\frac{\partial T}{\partial u}-\frac{2}{u+v}+\frac{\partial}{\partial u}\log \rho m\\
&\qquad+\frac{a'b'c'W}{\kappa\rho(u+v)^3}\left[\frac{\partial m}{\partial \theta}+\frac{\epsilon(u+v)^2\sqrt{\rho}}{2\sqrt{abc}}\left(D\frac{\partial l}{\partial\theta}+D'\frac{\partial m}{\partial\theta}\right)\right]=0,\\
&\frac{\partial T}{\partial v}-\frac{2}{u+v}+\frac{\partial}{\partial v}\log \rho l\\
&\qquad+\frac{a'b'c'W}{\kappa\rho(u+v)^3}\left[\frac{\partial l}{\partial \theta}+\frac{\epsilon(u+v)^2\sqrt{\rho}}{2\sqrt{abc}}\left(D'\frac{\partial l}{\partial\theta}+D''\frac{\partial m}{\partial\theta}\right)\right]=0,
\end{aligned}
$$

and the third is a consequence of these.

As the conditions of integrability of (35) are satisfied, we have the theorem: *When a transformation B_k of a surface S applicable to a hyperboloid is known, an infinitesimal deformation of S is given by a quadrature.*

As in the case of surfaces applicable to the paraboloids, we can give an intrinsic definition of surfaces applicable to a hyperboloid as follows:

Consider the two differential equations

$$
(38)\quad
\begin{aligned}
&l\left(\frac{\partial T}{\partial u}+\frac{\partial \log\rho}{\partial u}+\frac{\partial \log m}{\partial u}-\frac{2}{u+v}\right)\\
&\qquad+m\left(\frac{\partial T}{\partial v}+\frac{\partial \log\rho}{\partial v}+\frac{\partial \log l}{\partial v}-\frac{2}{u+v}\right)+\frac{\partial l}{\partial\theta}\frac{\partial\theta}{\partial u}+\frac{\partial m}{\partial\theta}\frac{\partial\theta}{\partial v}=0,\\
&l\left(\frac{\partial T}{\partial u}+\frac{\partial \log\rho}{\partial u}+\frac{\partial \log m}{\partial u}-\frac{2}{u+v}\right)\left(\frac{k(u+v)^2}{a'b'c'U}\frac{\partial\theta}{\partial v}-1\right)\\
&\qquad-m\left(\frac{\partial T}{\partial v}+\frac{\partial \log\rho}{\partial v}+\frac{\partial \log l}{\partial v}-\frac{2}{u+v}\right)\left(\frac{k(u+v)^2}{a'b'c'V}\frac{\partial\theta}{\partial u}-1\right)\\
&\qquad\qquad+\left(\frac{\partial m}{\partial\theta}\frac{\partial\theta}{\partial v}-\frac{\partial l}{\partial\theta}\frac{\partial\theta}{\partial u}\right)=0,
\end{aligned}
$$

in which ρ, l, m, U, V, W are given by (8), (11), (12), If T and θ are any solution of these equations, the functions D and D'', given by (34), and D',

given by either of the equivalent expressions

$$
(37)\quad
\begin{aligned}
&\frac{\partial T}{\partial u} - \frac{A}{l} + \frac{\partial \log \rho}{\partial u} - \frac{2}{u+v} + \frac{\partial \log m}{\partial u} + \frac{1}{l}\frac{\partial l}{\partial \theta}\frac{\partial \theta}{\partial u} - \frac{\epsilon \sqrt{abc}\, \rho^{\frac{3}{2}} A}{2(u+v)^2 Hl} D' = 0,\\
&\frac{\partial T}{\partial v} + \frac{A}{m} + \frac{\partial \log \rho}{\partial v} - \frac{2}{u+v} + \frac{\partial \log l}{\partial v} + \frac{1}{m}\frac{\partial m}{\partial \theta}\frac{\partial \theta}{\partial v} + \frac{\epsilon \sqrt{abc}\rho^{\frac{3}{2}} A}{2(u+v)^2 Hm} D' = 0,
\end{aligned}
$$

are the second fundamental coefficients of a surface applicable to the hyperboloid (1), the first fundamental coefficients being given by (2). Furthermore, when one has a surface S defined in this manner, one can obtain without quadratures the intrinsic equations of a second surface of the same kind, which is a transform under a B_k of S.

§ 13. *Continuous Deformation of S. Intrinsic Determination of Systems (Q) of Surfaces Applicable to a Hyperboloid. Systems (Q) of Ruled Surfaces.*

In view of the results of the preceding section we inquire under what conditions the equations

$$
(39)\quad
\begin{aligned}
&\frac{\partial x}{\partial u} = \sqrt{E} X_1, \qquad \frac{\partial X}{\partial v} = \sqrt{G} X_2,\\
&\frac{\partial x}{\partial w} = e^T\left[-(Fl+Gm)\sqrt{E}X_1 + (El+Fm)\sqrt{G}X_2 + \frac{2\epsilon\sqrt{abc}\rho^{\frac{3}{2}}A}{(u+v)^2} X_3\right]
\end{aligned}
$$

and similar ones in y and z define space referred to a triple system of surfaces, such that the surfaces $w =$ const. are applicable to one another and to the hyperboloid (1).

The discussion of this problem may be carried on as in § 4, with equations similar to (40), (42), (43) and (44), with the difference that now

$$
(41)\quad
\begin{aligned}
B &= 2\epsilon\sqrt{abc}\frac{d}{du}\left(\frac{\rho^{\frac{3}{2}}}{(u+v)^2}Ae^T\right) + [-(Fl+Gm)D + (El+Fm)D']e^T,\\
C &= 2\epsilon\sqrt{abc}\frac{d}{dv}\left(\frac{\rho^{\frac{3}{2}}}{(u+v)^2}Ae^T\right) + [-(Fl+Gm)D' + (El+Fm)D'']e^T.
\end{aligned}
$$

Proceeding as in § 4, we find that the expressions (41) must be equivalent to

$$
(49)\quad
\begin{aligned}
B &= \frac{a'\,b'\,c'}{k\,(u+v)^5} We^T\sqrt{\rho}\left[2\epsilon\sqrt{abc}\frac{\partial m}{\partial \theta} + \sqrt{\rho}\,(u+v)^2 J\left(\frac{D'}{l} - \frac{D}{m}\right)\right],\\
C &= \frac{a'\,b'\,c'}{k\,(u+v)^5} We^T\sqrt{\rho}\left[-2\epsilon\sqrt{abc}\frac{\partial l}{\partial \theta} + \sqrt{\rho}\,(u+v)^2 J\left(\frac{D''}{l} - \frac{D'}{m}\right)\right],
\end{aligned}
$$

which necessitates that the condition

$$
(50)\qquad El^2 + 2Flm + Gm^2 + \rho^2 A^2 = \frac{2a'\,b'\,c'\,W\rho}{k\,(u+v)^3} J
$$

be satisfied; that this is an identity is shown in § 14.

Also the following equations must be consistent with one another and must be satisfied:

$$
(52)\qquad
\begin{aligned}
\frac{\partial D}{\partial w} &= (DF - D'E)\, e^T + \frac{dB}{du} + \frac{2B}{u+v}, \\
\frac{\partial D'}{\partial w} &= (DG - D'F)\, e^T + \frac{dC}{du} - \frac{B}{2}\frac{\partial \log \rho}{\partial v} - \frac{C}{2}\frac{\partial \log \rho}{\partial u} \\
&= (D'F - D''E)\, e^T + \frac{dB}{dv} - \frac{B}{2}\frac{\partial \log \rho}{\partial v} - \frac{C}{2}\frac{\partial \log \rho}{\partial u}, \\
\frac{\partial D''}{\partial w} &= (D'G - D''F)\, e^T + \frac{dC}{dv} + \frac{2C}{u+v}.
\end{aligned}
$$

As in § 4 it can be shown readily that all the equations of condition of the problem, viz., (9), (14), (34), (52) and the Codazzi equations for S, are consistent with one another. Furthermore, when there exists a set of functions D, D', D'', θ, T satisfying these equations, equations similar to (53) § 5 are satisfied and hence there exists a triple system of surfaces of the kind sought, with respect to which the linear element of space has the form

$$
(54)\qquad
\begin{aligned}
ds^2 = {} & Edu^2 + 2Fdudv + Gdv^2 + 8He^T (ldv - mdu)\, dw \\
& + 4He^{2T}(El^2 + 2Flm + Gm^2 + \rho^2 A^2)\, dw^2.
\end{aligned}
$$

As regards the existence and determination of systems (Q) of this sort the analytical procedure is similar to that followed in § 5, and the result is the same, namely

There exist triple systems of surfaces such that the surfaces in one family are applicable to one another and to the hyperboloid.

With the aid of the results of Bianchi* one can readily extend the foregoing investigation and establish the existence of systems (Q) of surfaces applicable to any central quadric.

Thus far we have tacitly assumed that the surfaces S are not ruled, but by considerations similar to those of § 6 it may be shown that

There exist systems (Q) for which all of the surfaces $w = const.$ are ruled, and the determination of the intrinsic equations of such systems requires the integration of a system of partial differential equations of the first order.

In fact, for the case of the hyperboloid (1), we have

$$
(60)\qquad D = 0, \qquad D' = -2\frac{\sqrt{H}}{\rho}, \qquad D'' = 2(u+v)^2\sqrt{H}\varphi,
$$

* B., chapter 3.

where φ is independent of u. Now equations (14) reduce to

$$\frac{\partial \theta}{\partial u} = 0, \qquad \frac{\partial \theta}{\partial v} = \frac{a' b' c'}{k} V \varphi; \tag{61}$$

the first of equations (35) may be replaced by

$$T = \log \frac{(u+v)^2}{\rho m} f, \tag{67}$$

where f is independent of u and the second of (35) requires that f satisfy

$$\frac{\partial f}{\partial v} = f \frac{V'}{V}. \tag{68}$$

The first two of equations (52) are satisfied identically, and the last requires that φ in (60) be a solution of

$$\frac{\partial \varphi}{\partial w} + \frac{2abc}{k} f \frac{\partial \varphi}{\partial v} + \sigma f \varphi = 0, \tag{69}$$

where σ is given by

$$\sigma = \left(\frac{F}{m} + \frac{G}{l}\right) \frac{(u+v)^2}{\rho} - \frac{L_0}{lm^2} \rho A (u+v)^2 + \frac{2abc}{k} \left(\frac{2}{u+v} + \frac{V'}{V}\right). \tag{69'}$$

It can readily be shown that each set of solutions of equations (61), (68) and (69) gives a system (Q) of the kind sought.

§ 14. *Conjugate Systems. Transformation of Systems* (Q).

If the equation

$$\bar{x} = x + l \frac{\partial x}{\partial u} + m \frac{\partial x}{\partial v} \tag{70}$$

be differentiated, we obtain

$$\begin{aligned} \frac{\partial \bar{x}}{\partial u} &= L_0 \frac{\partial x}{\partial u} + M_0 \frac{\partial x}{\partial v} + \left(\frac{\partial l}{\partial \theta} \frac{\partial x}{\partial u} + \frac{\partial m}{\partial \theta} \frac{\partial x}{\partial v}\right) \frac{\partial \theta}{\partial u} + (Dl + D'm) X_3, \\ \frac{\partial \bar{x}}{\partial v} &= P_0 \frac{\partial x}{\partial u} + Q_0 \frac{\partial x}{\partial v} + \left(\frac{\partial l}{\partial \theta} \frac{\partial x}{\partial u} + \frac{\partial m}{\partial \theta} \frac{\partial x}{\partial v}\right) \frac{\partial \theta}{\partial v} + (D'l + D''m) X_3. \end{aligned} \tag{71}$$

The affine transformation of Ivory which establishes the correspondence of applicability of S and $\bar{S}$ is given analytically by*

$$\begin{aligned} \bar{u} &= \frac{\frac{b}{b'}\left[\frac{a}{a'}(1-uv) - \frac{c}{c'}(1+uv)\right](1+\sin\theta) + \frac{ac}{a'c'}\left[(u-v) + \frac{b}{b'}(u+v)\right]\cos\theta}{\frac{b}{b'}\left[\frac{a}{a'}(1-uv) + \frac{c}{c'}(1+uv)\right]\cos\theta + \frac{ac}{a'c'}\left[(u-v) - \frac{b}{b'}(u+v)\right](1+\sin\theta)}, \\ \bar{v} &= \frac{1-\sin\theta}{\cos\theta}. \end{aligned} \tag{72}$$

* B., p. 60.

From (71) and (72) we have accordingly

$$\frac{\partial \bar{x}}{\partial \bar{u}} = \frac{(u+v)\, a'\, b'\, c'\, \psi^2}{2Wabc\,(1+\sin\theta)}\left[\frac{L_0}{m}\frac{\partial x}{\partial u} + \frac{M_0}{m}\frac{\partial x}{\partial v} - \frac{2\epsilon\sqrt{abc}}{(u+v)^2\sqrt{\rho}}X_3\right],$$

$$(77)\quad \frac{\partial \bar{x}}{\partial \bar{v}} = -\,(1+\sin\theta)\left[\left(\frac{J}{m} - \frac{L_0\, k\rho\,(u+v)^2}{2a'\, b'\, c'\, V}\right)\frac{\partial x}{\partial u}\right.$$

$$\left. + \left(\frac{J}{l} - \frac{M_0\, k\rho\,(u+v)^2}{2a'\, b'\, c'\, V}\right)\frac{\partial x}{\partial v} + \frac{\epsilon\,(u+v)\, k\sqrt{abc}\sqrt{\rho}}{Wa'\, b'\, c'}X_3\right],$$

where ψ denotes the denominator of $\bar{u}$ in (72).

Proceeding as in § 7 we find that

$$(84)\qquad J = -\frac{2abc\,(u+v)\, lm}{a'\, b'\, c'\, W},$$

and we establish the identity (50) in the form

$$(85)\qquad El^2 + 2Flm + Gm^2 + \rho^2 A^2 = -\frac{4abclm\rho}{k\,(u+v)^2}.$$

Furthermore it can be shown that (70) may be written in the form

$$(92)\qquad x = \bar{x} + \bar{l}\frac{\partial \bar{x}}{\partial \bar{u}} + \bar{m}\frac{\partial \bar{x}}{\partial \bar{v}},$$

where

$$(94)\qquad \bar{l} = \frac{(1+\sin\theta)\, k\rho W\,(u+v)}{2a'\, b'\, c'\, \psi^2}, \qquad \bar{m} = -\frac{a'\, b'\, c'\, W}{2abc\,(1+\sin\theta)\,(u+v)}.$$

From these expressions we have

$$(95)\qquad \rho = -\frac{4abc\bar{l}\bar{m}}{k\,(\bar{u}+\bar{v})^2}$$

which is analogous to the expression

$$(83)\qquad \bar{\rho} = -\frac{4abclm}{k\,(u+v)^2},$$

arising in the proof of (85).

One shows readily that (72) is reciprocal in form, and by means of the expressions derived therefrom for u and v in terms of $\bar{u}$ and $\bar{v}$ it can be proved that

If $\bar{S}$ is a transform of S by a transformation B_k, S may be obtained from $\bar{S}$ by another $\bar{B}_k$, which is the inverse of the former.

When a system (Q) is transformed by a B_k, whose equations are

$$(96)\qquad x_1 = x + l_1\frac{\partial x}{\partial u} + m_1\frac{\partial x}{\partial v},$$

where l_1 and m_1 denote the result of replacing θ and k in l and m by θ_1 and k_1, we obtain a system of surfaces S_1 applicable to the same hyperboloid. If each surface S_1 undergoes a transformation of parameters to a new set u_1 and v_1, analogous to $\bar{u}$ and $\bar{v}$, defined by (72), we have space referred to a triple system of surfaces of parameters u_1, v_1, w.

As shown in § 9, the direction of the tangent to the curves of parameter w are given by

$$\frac{\partial x_1}{\partial w} = X_1 \sqrt{E} e^T [F(l_1 - l) + G(m_1 - m)]$$

$$(98) \qquad - X_2 \sqrt{G} e^T [E(l_1 - l) + F(m_1 - m)]$$

$$+ X_3 \left[e^T \frac{2\epsilon \rho^{\frac{3}{2}} \sqrt{abc} A}{(u+v)^2} + l_1 B + m_1 C - \frac{2\epsilon_1 k_1 (u+v) \sqrt{\rho} \sqrt{abc}}{a_1' b_1' c_1' W_1} \frac{\partial \theta_1}{\partial w} \right]$$

When we take (92) in place of (96), this reduces to

$$(99) \qquad \frac{\partial \bar{x}}{\partial w} = 2X_3 \left[\frac{abc\rho e^T}{k(u+v)^2} (Dl^2 - D'' m^2) - \frac{\epsilon k (u+v) \sqrt{\rho} \sqrt{abc}}{a' b' c' W} \frac{\partial \theta}{\partial w} \right].$$

Hence the tangents to the curves of parameter w are parallel to the corresponding normals to the surfaces S, and consequently, if we say that the surfaces $\bar{S}$ form a system *conjugate* to the given system, we have the theorem:

The system conjugate to a given system (Q) is a system (Q).

As regards the existence of a generalized transformation of a system (Q) into a system (Q) the situation is similar to that set forth in § 10. However, in the next section we give an example of particular systems (Q) for which such transformations have been established.

§ 15. *Isogonal Deformations of Pseudospherical Surfaces.*

If in (1) we put*

$$(109) \qquad a = b = i, \qquad c = 1$$

we have that Q is the imaginary sphere $x^2 + y^2 + z^2 + 1 = 0$, and consequently S is a pseudospherical surface. In this case we have

$$(110) \qquad E = G = 0, \qquad F = \frac{2}{(u+v)^2}, \qquad H = \frac{-1}{(u+v)^4}, \qquad \rho = 1.$$

* Cf. B., §§ 49, 50.

Taking

$$k = \cos^2 \sigma, \tag{111}$$

we find

$$a' = b' = i \sin \sigma, \qquad c' = \sin \sigma,$$

$$l = \frac{1 - \sin \sigma}{2}(u + v)\frac{\lambda}{\mu}, \qquad m = \frac{1 + \sin \sigma}{2}(u + v)\frac{\mu}{\lambda}, \tag{112}$$

$$W = \frac{2}{\sin^2 \sigma} \frac{\lambda\mu}{1 + \sin \theta},$$

where

$$\lambda = \cos \theta + u(1 + \sin \theta), \qquad \mu = \cos \theta - v(1 + \sin \theta). \tag{113}$$

If we put

$$u = \alpha + i\beta, \qquad v = \alpha - i\beta, \qquad e^{i\phi} = \frac{1 - \sin \sigma}{\cos \sigma} \frac{\lambda}{\mu}, \tag{114}$$

we have for the new fundamental coefficients of the surface

$$E' = G' = \frac{1}{\alpha^2}, \qquad F' = 0,$$

$$\Delta = 2D' + D + D'', \qquad \Delta' = i(D - D''), \qquad \Delta'' = 2D' - D - D''. \tag{115}$$

It follows that

$$l = \alpha \cos \sigma e^{i\phi}, \qquad m = \alpha \cos \sigma e^{-i\phi}, \qquad A = -\sin \sigma. \tag{116}$$

Equations (14) become

$$\begin{aligned} \frac{\partial \varphi}{\partial \alpha} &= \frac{\sin \varphi}{\alpha \cos \sigma} + \alpha \tan \sigma (\Delta \cos \varphi + \Delta' \sin \varphi), \\ \frac{\partial \varphi}{\partial \beta} &= \frac{\cos \sigma - \cos \varphi}{\alpha \cos \sigma} + \alpha \tan \sigma (\Delta' \cos \varphi + \Delta'' \sin \varphi). \end{aligned} \tag{117}$$

Furthermore (39) may be replaced by

$$\begin{aligned} &\frac{\partial x}{\partial \alpha} = \frac{X_1'}{\alpha}, \qquad \frac{\partial x}{\partial \beta} = \frac{X_2'}{\beta}, \\ &\frac{\partial x}{\partial w} = h[\cos \sigma (\sin \varphi X_1 - \cos \varphi X_2) + \sin \sigma X_3], \end{aligned} \tag{118}$$

where

$$h = -\frac{2ie^T}{(u+v)^2} = -\frac{ie^T}{2\alpha^2}. \tag{119}$$

Equations (118) define a system (Q) such that the surfaces $w =$ const. are pseudospherical and the curves of parameter w are isogonal trajectories of these surfaces. These are the systems which Bianchi has discovered and, in fact, the equations are in the form given by him.*

For this case the equation analogous to (103), Part I, becomes

$$h_{33} = \frac{\alpha^2}{\cos^2\sigma}(h_{13}^2 + h_{23}^2), \tag{120}$$

when the linear element of space is written

$$ds^2 = \frac{d\alpha^2 + d\beta^2}{\alpha^2} + 2h_{13}\, d\alpha dw + 2h_{23}\, d\beta dw + h_{33}\, dw^2. \tag{121}$$

Bianchi* shows that the relation (120) characterizes systems (Q) for which the curves of parameter w are isogonal trajectories of the pseudospherical surfaces $w =$ const. When Bianchi expresses the condition (120) for the system arising from a given system (Q) by means of a transformation B_{σ_1}, he finds that φ_1 must satisfy in addition to (117), in which σ is replaced by σ_1, the further equation

$$\begin{aligned} \frac{\partial \varphi_1}{\partial w} &= \frac{\alpha \tan\sigma_1}{\sin\sigma}\cos\varphi_1 \frac{\partial h}{\partial u} + \frac{u\tan\sigma_1}{\sin\sigma}\frac{\partial h}{\partial v} \\ &+ \frac{h\sin\sigma_1}{\sin\sigma - \sin\sigma_1} - \frac{h\cos\sigma\sin^2\sigma_1}{\sin\sigma\cos\sigma_1(\sin\sigma - \sin\sigma_1)}\cos(\varphi_1 - \varphi). \end{aligned} \tag{122}$$

Because of the particular form of the functions Bianchi was enabled to factor the equation analogous to (104) § 10, but we have indicated a method by which π can be formed, even if the factors are not apparent.

Thus we have a particular case in which a system (Q) of surfaces applicable to a central quadric is transformable into another system (Q). This strengthens the belief stated at the end of the preceding section. Furthermore, we are of the opinion that (104) is the sufficient as well as necessary condition that a family of continuous deforms of a central quadric forms a system (Q).

It should be added here that by suitable changes in constants and variables it is possible to transform the results of the preceding sections so that one shall obtain similar theorems for surfaces applicable to any central quadric.

* L. c., p. 20.

§ 16. *Deformation of the Hyperboloid of Revolution and of Bertrand Curves.*

We shall close our discussion with the consideration of the continuous deformation of a ruled-surface applicable to an hyperboloid of revolution, because one derives therefrom a continuous array of applicable curves of Bertrand.

Let S be a ruled surface applicable to an hyperboloid of revolution, then the line of striction is a Bertrand curve and corresponds to the circle of gorge on the hyperboloid. Referring to (1), we see that in the present case $b = a$ and the line of striction is given by $uv - 1 = 0$. Accordingly we effect the change of parameters

$$u = \frac{1-t}{v_1}, \qquad v = v_1,$$

so that the curve $t = 0$ on each surface S is the line of striction.

In this case

$$E_1 = \Sigma\left(\frac{\partial x}{\partial t}\right)^2 = \frac{E}{v^2}, \qquad F_1 = \Sigma \frac{\partial x}{\partial t}\frac{\partial x}{\partial v_1} = \frac{u}{v^2}E - \frac{1}{v}F, \quad G_1 = G - \frac{2u}{v}F + \frac{u^2}{v^2}E.$$

The direction-cosines of the tangent to the curve $t = 0$ are

$$\alpha, \beta, \gamma = \frac{\sqrt{a^2+c^2}}{2a}[(X_2 - X_1), \quad (Y_2 - Y_1), \quad (Z_2 - Z_1)],$$

where $X_1, \cdots, Z_2$ are given by (39). The direction-cosines of the principal normal are X_3, Y_3, Z_3 and of the binomial

$$\lambda, \mu, \nu = \frac{\sqrt{a^2+c^2}}{2c}[X_2 + X_1, Y_2 + Y_1, Z_2 + Z_1].$$

We denote by $\bar{S}$ the locus of the Bertrand curves $t = 0$ on the surfaces $w =$ const. of a system (Q) of the kind considered. The direction-cosines of the normal to $\bar{S}$ are of the form

$$\bar{X} = r[c\sqrt{a^2+c^2}A\lambda - a(\sqrt{E}l - \sqrt{G}m)X_3],$$

where

$$r = \sqrt{(\sqrt{E}l + \sqrt{G}m)^2 a^2 + c^2(a^2+c^2)A^2}.$$

If $\bar{\omega}$ denotes the angle between the principal normal to $t = 0$ and the normal to the surface, then

$$\cos\bar{\omega} = ar(\sqrt{G}m - \sqrt{E}l), \qquad \sin\bar{\omega} = crA\sqrt{a^2+c^2}.$$

One finds without difficulty the first and second fundamental coefficients of $\bar{S}$, and therefrom obtains

$$\frac{\cos\bar{\omega}}{\rho_0} = \frac{\bar{D}}{\bar{E}} = \frac{1}{2}[\,c\,(1+v^2)^2\,\varphi + 2\,](\,\sqrt{G}m - \sqrt{E}l\,)\,r,$$

$$\frac{1}{T} = \frac{\bar{F}\bar{D} - \bar{E}\bar{D}'}{\bar{E}\sqrt{\bar{E}\bar{G} - \bar{F}^2}} = -\,cr^2\,(a^2+c^2)\,A$$

$$+\,r^2\sqrt{a^2+c^2}\left[\frac{e^{-T}}{4acv^2}\left(C - \frac{B}{v^2}\right) - \frac{(lv^2+m)c}{(1+v^2)^2}\left(c\varphi + \frac{2}{(1+v^2)^2}\right)\right]$$

$$(\,\sqrt{E}l - \sqrt{G}m\,)\,(1+v^2)^3,$$

where ρ_0 and T are the radii of first curvature and geodesic torsion respectively of the curve $t = 0$ and B and C are the functions defined by (41). Since the torsion is given by

$$\frac{1}{\tau} = \frac{1}{T} + \frac{d\bar{\omega}}{ds_v},$$

we are in a position to find τ. By easy reduction it is found that

$$\frac{1}{\tau} = \frac{1}{2}(1+v^2)\varphi.$$

Since the element of any of the curve is given by $\sqrt{E}\,dv_1$, one finds readily that

$$v = \tan\frac{s}{2a}.$$

Hence the intrinsic equations of the curve are

$$(123)\qquad \frac{1}{\rho_0} = \frac{1}{2a}\left(c\varphi\sec^4\frac{s}{2a} + 2\right), \qquad \frac{1}{\tau} = \frac{\varphi}{2}\sec^4\frac{s}{2a},$$

where from § 13 it follows that φ satisfies the equations

$$(124)\qquad \begin{gathered} \frac{\partial\theta}{\partial s} = \frac{(a^2+k)\sqrt{c^2-k}\,2aV\varphi}{k\,(s^2+4a^2)}, \qquad \frac{\partial f}{\partial s} = \frac{f}{V}\frac{\partial V}{\partial v}\cdot\frac{2a}{s^2+4a^2}, \\ \frac{\partial\varphi}{\partial w} + \frac{ac}{k}f\,(s^2+4a^2)\frac{\partial\varphi}{\partial s} + \sigma f\varphi = 0, \end{gathered}$$

σ being a function of s and θ obtained by replacing u and v in (69′) by $\cot s/2a$ and $\tan s/2a$ respectively.

The surfaces $w =$ const. of the system (Q) conjugate to the given system are likewise applicable to the same hyperboloid. Moreover to a curve of Bertrand of the original system corresponds a curve of Bertrand of the second system in such a way that the join of corresponding points is tangent to the two surfaces of conjugate systems; we call the second curve the *conjugate Bertrand curve*. Hence we have the theorem:

To each function φ *satisfying equations* (124) *there corresponds a family of curves of Bertrand defined by* (123) *such that all of these curves are continuous deforms of one another, the parameter of deformation being* w, *the direction of deformation being parallel to the principal normal of the conjugate curve at the corresponding point.*

Later we shall consider general deformations of Bertrand curves.

PRINCETON,
November 15, 1912.